Lecture 2: Amortized Analysis & Splay Trees

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Overview

- Introduction
  - Splay Trees
- Implementing Splay-Trees
  - Setup
  - Rotations & Splay Operation
  - Analysis
- Conclusion & Open Problems
- Acknowledgements

Why Splay Trees?

Binary search trees:

- extremely useful data structures (pervasive in computer science/industry)
- worst-case running time per operation $\Theta(\text{height})$
- Need technique to balance height.
- Different implementations: red-black trees [CLRS 2009, Chapter 13], AVL trees [CLRS 2009, Exercise 13-3] and many others (see [CLRS 2009, Chapter notes of ch. 13].
- All these implementations are quite involved, require extra information per node (i.e. more memory) and difficult to analyze.

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Splay trees are:

- Easier to implement
- don't keep any balance info!

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Theorem ([Sleator & Tarjan 1985])

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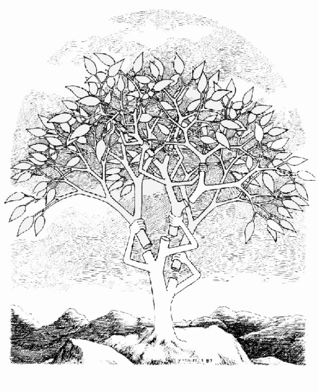

A Self-Adjusting Search Tree

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How do we fix this? By adding different kinds of rotations!

Basic Rotations

Rotation type 1: *zig-zag rotations*

Basic Rotations (continued)

Rotation type 2: *zig-zig rotations*

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Rotation type 3: *normal rotations (zigs)*

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  - If node of k in tree is a child of the root, perform normal rotation (zig).

Example

Example (continued)

Setup

Notation:

- $n \leftarrow$ number of elements (we denote the elements by $1, 2, \ldots, n$)
- $m \leftarrow$ number of operations. That is

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- $SEARCH(k) \leftarrow$ find whether element k is in tree
- $INSERT(k) \leftarrow$ insert element k in our tree
- $DELETE(k) \leftarrow$ delete element k from our tree

Splay Tree Algorithm

Input: set of elements $\{1, 2, \ldots, n\}$

Output: at each step, a binary-search tree data structure and the answer to the query being asked.

1. $SEARCH(k) \to$ after searching for k, if k in the tree, do $SPLAY(k)$. If k not in tree, do $SPLAY(k')$ where k' is the last node seen in the traversal
2. $INSERT(k) \to$ standard insert operation, then do $SPLAY(k)$
3. $DELETE(k) \to$ standard delete operation, then $SPLAY(parent(k))$
    - delete first "moves k to the bottom of tree" (by finding successor)
    - then delete k as in the cases where k has at most one child
    - then we splay the parent of k (after we place k at the bottom)
    - see [CLRS 2009, Chapter 12] for a recap (and correct implementation)

Figure: Is that it?

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So long as $\Phi(D_m) \geq \Phi(D_0)$ then amortized charge is an upper bound on amortized cost.

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Examples (max potential):

Example - min potential

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Lemma (Amortized cost from SPLAY Subroutines)

The charge γ of an operation (zig, zig-zig, zig-zag) is bounded by:

$$\gamma \leq \begin{cases} 3 \cdot (\text{rank}'(k) - \text{rank}(k)) & \textit{for zig-zig, zig-zag} \\ 3 \cdot (\text{rank}'(k) - \text{rank}(k)) + 1 & \textit{for zig} \end{cases}$$

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Lemma (Total Amortized Cost of $SPLAY(k)$)

Let T be our current tree, with root t and k be a node in this tree. The charge of $SPLAY(k)$ is

$$\leq 3 \cdot (\text{rank}(t) - \text{rank}(k)) + 1 \leq 3 \cdot \text{rank}(t) + 1 = O(\log n)$$

Proof of First Lemma (charge to zig)

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Proof of Second Lemma (total charge of $SPLAY(k)$)

Analysis - Amortized cost

1. For each operation (INSERT, SEARCH, DELETE) we have:[1]

$$\begin{aligned}\text{(charge per operation)} &= \text{(charge of SPLAY)} \\ &+ \text{(potential change \textit{not} from SPLAY)}\end{aligned}$$

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    3. *INSERT* $\rightarrow$ adding new element k increases ranks of all ancestors of k post insertion (might be $O(n)$ of them)

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Handling INSERT potential

Let us check the potential change after an insert:

Final Analysis

After Learning Splay Trees

Figure: You to whoever taught you red-black trees

Conclusion

- Splay trees gives us a fairly *simple algorithm* to balance a tree
- Great amortized cost!

$$O(\log n) \quad \text{per operation}$$

- Analysis is very clever (yet principled!)
- Remember: this only works in the amortized setting (may be very bad for client-server model for instance)

Dynamic Optimality Conjecture

Open Question ([Sleator & Tarjan 1985])

Splay Trees are optimal (within a constant) in a very strong sense:

Given a sequence of items to search for $a_1, \ldots, a_m$*, let* OPT *be the minimum cost of doing these searches + any rotations you like on the binary search tree.*
You can charge 1 for following tree pointer (parent $\rightarrow$ *child or child* $\rightarrow$ *parent), charge 1 per rotation.*

Conjecture: *Cost of splay tree is* $O(OPT)$*.*

Note that for OPT, you get to look at the sequence of searches first and plan ahead. (we will cover this in more detail in the online algorithms part of the course)
Also, OPT can adjust the tree so it's even better than the static optimal binary search trees you may have seen in CS 341.

Acknowledgement

- Lecture based largely on Anna Lubiw's notes. See her notes at

  https://cs.uwaterloo.ca/~r5olivei/courses/2025-spring-cs466/lectures-info/anna-lubiw-splay-trees.pdf
- Picutre of self-adjusting tree taken from Robert Tarjan's website

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