PROBLEM 1 (30 Points) - PIT

In this question we will devise a different (and less efficient) hitting set generator for sparse polynomials. This hitting set generator works for some other circuit classes for which we don't know that the Klivans-Spielman hitting set works, for instance, read-once formulas and k-sums of read-once formulas.

Shpilka-Volkovich (SV) generator: Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be distinct numbers. For $i \in [t]$, let $\mathcal{G}_i : \mathbb{C} \to \mathbb{C}^n$ be defined as follows:

$$\mathcal{G}_i(y_i) := \left(\frac{\prod_{j=1}^n (y_i - \alpha_j)}{y_i - \alpha_1}, \ldots, \frac{\prod_{j=1}^n (y_i - \alpha_j)}{y_i - \alpha_n} \right)$$

and finally, let the SV generator $\mathcal{G}^{(t)} : \mathbb{C}^{2t} \to \mathbb{C}^n$ be defined as

$$\mathcal{G}^{(t)}(y_1, \ldots, y_t, z_1, \ldots, z_t) := z_1 \cdot \mathcal{G}_1(y_1) + \cdots + z_t \cdot \mathcal{G}_t(y_t)$$

Prove that $\mathcal{G}^{(1+2\log s)}$ hits $\Sigma\Pi$ circuits of size s, where you can assume $s \geq n$.

Hint 1: if $P(\mathbf{x})$ is a non-zero s-sparse polynomial, show that there is a variable x_i such that at most half of the monomials of P have the same power of x_i.

Hint 2: the SV generator has the ability to "select" certain variables ($\mathcal{G}^{(t)}$ can select up to t variables).

PROBLEM 2 (40 Points) - PIT

In this question, we will show that the SV generator is also a hitting set generator for the class of $\texttt{hom-}\Sigma \wedge \Sigma$ circuits. More precisely, we will show that $\mathcal{G}^{(2\log s)}$ hits $\texttt{hom-}\Sigma \wedge \Sigma$ circuits of size s.

That is, if $P(\mathbf{x})$ is a nonzero polynomial computed by a $\texttt{hom-}\Sigma \wedge^d \Sigma$ circuit of size s (where size is the top fanin of the circuit multiplied by d), that is,

$$P(\mathbf{x}) = \sum_{i=1}^{s} \ell_i(\mathbf{x})^d$$

then $P \circ \mathcal{G}^{(2\log s)} \not\equiv 0$

To show the above result, we will use our knowledge of the lower bounds for the class $\texttt{hom-}\Sigma \wedge \Sigma$ which we learned in the previous homework.

1. Use the dimension of partial derivatives measure to show that if P is a non-zero polynomial computed by a $\texttt{hom-}\Sigma \wedge \Sigma$ circuit of size s, then the *leading monomial* of P must have support $\leq \log s$
2. after proving part 1, use the SV generator to hit P

PROBLEM 3 (30 Points) - Reconstruction

Give a deterministic, $\mathsf{poly}(nds)$-time algorithm to reconstruct a polynomial in $\Sigma^s\Pi^d$ on n variables, given only black-box access to it.

Hint: Klivans-Spielman