

Lecture 22: Hardness of Approximation

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Overview

- Background and Motivation
 - Why Hardness of Approximation?
 - How do we prove Hardness of Approximation?
 - Hardness of Approximation - Example
- Proofs & Hardness of Approximation
- Conclusion
- Acknowledgements

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- Important to know the limits of efficient algorithms!

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- For hardness of approximation what we would like is a (more robust) reduction of the form:
 - maps every YES instance of L to a YES instance of \mathcal{C}
 - maps every NO instance of L to a VERY-MUCH-NO instance of \mathcal{C}

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 - Efficient route planning (mail system, shuttle bus pick up and drop off...)
- One of the famous NP-complete problems

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- 3 In our case, let's reduce it to the *Hamiltonian Cycle Problem*

Theorem

If there is an algorithm M which solves TSP without repetitions with α -approximation, then $P = NP$.

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- 5 Thus, M on input H will output a Hamiltonian Cycle of G , if G has one, or it will output a solution with value $\geq (1 + \alpha) \cdot |V|$

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Complexity Classes

- **NP:** Set of languages $L \subseteq \{0, 1\}^*$ such that there exists a poly-time Turing Machine V , such that:

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- **co-RP:** languages $L \subseteq \{0, 1\}^*$ s.t. $\bar{L} \in RP$

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Theorem (PCP theorem [AS'98, ALMSS'98])

$$PCP[\log n, 1] = NP$$

PCP and Approximability of Max 3SAT

Definition (Max 3SAT)

- **Input:** a 3CNF formula φ on boolean variables x_1, \dots, x_n and m clauses
- **Output:** the maximum number of clauses of φ which can be simultaneously satisfied.

Theorem

- 1 *The PCP theorem implies that there is an $\varepsilon > 0$ such that there is no polynomial time $(1 + \varepsilon)$ -approximation algorithm for Max 3SAT, unless $P = NP$.*
 - 2 *Moreover, if Max 3SAT is hard to approximate within a factor of $(1 + \varepsilon)$, then the PCP theorem holds.*
- In other words, the PCP theorem and the hardness of approximation of Max 3SAT are equivalent.

PCP and Approximability of Max 3SAT

- 1 Let us assume the PCP theorem holds.
 - Let $L \in PCP[\log n, 1]$ be an NP-complete problem.
 - Let V be the $(O(\log n), q)$ verifier for L , where q is a constant

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 - $x \in L \Rightarrow \varphi_x$ is satisfiable
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 - $x \in L \Rightarrow \varphi_x$ is satisfiable
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- 4 Enumerate all random inputs R for the verifier V .
 - Length of each random string is $O(\log n)$, by definition. So number of such random inputs is $\text{poly}(n)$.
 - For each R , V chooses q positions i_1^R, \dots, i_q^R and a boolean function $f_R : \{0, 1\}^q \rightarrow \{0, 1\}$ and accepts iff $f_R(w_{i_1^R}, \dots, w_{i_q^R}) = 1$.

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- 2 Simulate the computation f_R of the verifier for different random inputs R and witnesses w as a Boolean formula.
 - Can be done with a CNF of size 2^q
 - Converting to 3CNF we get a formula of size $q \cdot 2^q$

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- 4 If $x \in L$ then there is a witness w such that $V(x, w)$ accepts for every random string R . In this case, φ_x is satisfiable!
- 5 If $x \notin L$ then the verifier says NO for half of the random strings R .
 - For each such random string, at least one of its clauses fails
 - Thus at least $\varepsilon = \frac{1}{2 \cdot q \cdot 2^q}$ of the clauses of φ_x fails.

Conclusion

- Important to study hardness of approximation for NP-hard problems
- Different hard problems have different approximation parameters
- For hardness of approximation, need more *robust reductions* between combinatorial problems
- Proof systems, in particular *Probabilistic Checkable Proofs*, allows us to get such strong reductions
- Many more applications in computer science and industry!
 - Program Checking (for software engineering)
 - Zero-knowledge proofs in cryptocurrencies
 - many more...

Acknowledgement

- Lecture based largely on:
 - Section's 1-3 of Luca's survey [Trevisan 2004]
 - [Motwani & Raghavan 2007, Chapter 7]
- See Luca's survey <https://arxiv.org/pdf/cs/0409043>

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