Lecture 19: Streaming

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Overview

- Introduction
  - Data Streaming
  - Basic Examples
- Main Examples
  - Heavy hitters
  - Distinct Elements
  - Weighted Heavy Hitters
- Acknowledgements

Why streaming?

In today's world we have to deal with *big data*. But not all big data are created equal. Today we will study one way in which massive data can appear in our lives: *streaming*.

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How can we deal with it/model it? What can we do if we cannot even see the whole input?

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Goal: minimize space complexity (in bits) and the processing time.

Examples of Streaming Problems

Example (Sum of elements)

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- **Task:** maintain the current median of elements we have seen so far

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Example (Heavy hitters)

- **Input stream:** $a_1, \dots, a_N$ integers from $[-2^b + 1, 2^b - 1]$, $\epsilon > 0$
- **Task:** maintain set of elements that contains elements that have appeared more than ϵ-fraction of the time (a.k.a. *heavy hitters*)
- **Constraint:** allowed to also output *false positives* (low hitters), but not allowed to miss any heavy hitter!

Majority Element - Algorithm

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- At end of stream, return element in S_N

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- Space used: $O(b + \log N)$ (stored set S_t which has at most one element and counter)

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6. Return the array T with the counter array C

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- For element $e \in \Sigma$, let $est(e) = \begin{cases} C[j], & \text{if } e = T[j] \\ 0, & \text{otherwise.} \end{cases}$

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- If we don't increase $est(e)$ by 1 when we see an update to e then we decrement k counters and discard current update to e
- So we drop $k + 1$ distinct stream updates, but there are N updates, so we won't increase $est(e)$ by 1 (when we should) at most $\frac{N}{k+1} \leq \epsilon N$ times.

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  - $est(e) > 0 \Rightarrow e$ is in T
  - Space used is $O(k \cdot (\log(\Sigma) + \log N)) = O((1/\epsilon) \cdot (b + \log N))$ bits

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Example (Distinct elements)

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  - If we know that t^{th} smallest value is T, then $T \approx \frac{tm^3}{D} \Rightarrow D \approx \frac{tm^3}{T}$

Distinct Elements - algorithm

- Choose a random hash function h from strongly 2-universal hash family
- For each item a_i in the stream:
  - Compute $h(a_i)$
  - update list that stores the t smallest hash values
  - After all data has read, let T be t^{th} smallest hash value in data stream.

$$\text{Return } Y = \frac{tm^3}{T}$$

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- At least t hash values smaller than $\frac{(1-\epsilon/2)\cdot tm^3}{D}$
- Random variable $X_i = \begin{cases} 1, & \text{if } h(a_i) \leq \frac{(1-\epsilon/2)\cdot tm^3}{D} \\ 0, & \text{otherwise} \end{cases}$

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- but we assumed we have at least t such elements! Now need to show that this cannot happen with high probability

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  - Chebyshev's inequality:

$$\begin{aligned}\Pr[X > t] &= \Pr[X > t\cdot(1-\epsilon/2) + \epsilon\cdot t/2] \\ &\le \Pr[|X - \mathbb{E}[X]| > \epsilon\cdot t/2] \le \frac{4\cdot \mathrm{Var}[X]}{\epsilon^2 t^2} \le \frac{4}{\epsilon^2 t}\end{aligned}$$

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Similar calculation as previous slide.[1]
Practice problem: do this part of the proof.

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- $\Pr[Y > (1+\epsilon)\cdot D] \le \dfrac{4}{\epsilon^2 t}$
- $\Pr[Y < (1-\epsilon)\cdot D] \le \dfrac{4}{\epsilon^2 t}$
- Setting $t = 24/\epsilon^2$ gives us

$$\Pr[(1-\epsilon)\cdot D \le Y \le (1+\epsilon)\cdot D] \ge 1 - \frac{8}{\epsilon^2 t} = 2/3$$

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  - compute hash in $O(\log m)$ time
  - Since we keep track of $O(1/\epsilon^2)$ elements, and need to update the list, this takes $O(1/\epsilon^2)$ time (though there are smarter ways)

Heavy hitters with weights

Example (Weighted heavy hitters)

- **Input stream:** $(a_1, w_1), \ldots, (a_N, w_N)$ tuples of integers from $\Sigma = [-2^b + 1, 2^b - 1]$, parameter $q \in \mathbb{N}$

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- **Constraint:** allowed to also output *false positives* (low hitters), but not allowed to miss any heavy hitter!

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- Let's maintain $k \cdot \ell$ counters $C_{i,j}$, where each $C_{i,j}$ adds the weight of items that are mapped to j^{th} entry by the i^{th} hash function. Start with $C_{i,j} = 0$ for all $1 \leq i \leq k$ and $1 \leq j \leq \ell$.

Weighted heavy hitters - algorithm

- Given (a_t, w_t), for each $1 \leq i \leq k$ set $C_{i,h_i(a_t)} \leftarrow C_{i,h_i(a_t)} + w_t$.
- At the end,[2] report all elements e with

$$\min_{1 \leq i \leq k} C_{i,h_i(e)} \geq q$$

- Data structure as a table:

[2] In this version need to do second pass over data. But this can be fixed. Practice problem: fix this so that we can report on the fly.

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  - Hash functions h_i chosen independently $\Rightarrow$

$$\Pr\left[\min_{1 \leq i \leq k} Z_i \geq \epsilon \cdot Q\right] \leq \left(\frac{1}{\epsilon \ell}\right)^k$$

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- Space requirement for counters $O(1/\epsilon\cdot\log(1/\delta))$
- Space required to store all hash functions and evaluation time $O(k\cdot\ell)$

Acknowledgement

- Lecture based largely on Lap Chi's notes and David Woodruff's notes.
- See Lap Chi's notes at
  https://cs.uwaterloo.ca/~lapchi/cs466/notes/L05.pdf
- See David's notes at
  https://www.cs.cmu.edu/~15451-s20/lectures/lec6.pdf