Lecture 17: Online Algorithms & Paging

Rafael Oliveira

University of Waterloo Cheriton School of Computer Science rafael.oliveira.teaching@gmail.com

June 19, 2024

Overview

- Part I
 - Why Study Online Algorithms?
 - Competitive Analysis
 - Examples
- Paging & Caching
- Conclusion
- Acknowledgements

Why Study Online Algorithms?

• Online algorithms are important for many applications, when we need to make decisions right when we receive the information.

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 - Stock Market
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 - Machine Learning (regret minimization)
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- Competitive Analysis: measures performance of our algorithm against best algorithm that could see into the future (that is, see the entire input beforehand)¹
 - Worst-case analysis



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- Today, we will only see algorithms which must deal with the input as it receives it, *no constraints in memory*.
 - Goal here is to be competitive against any offline algorithm (that is, algorithms that could see the entire input beforehand)
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Definition (Randomized Competitive Ratio)

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- Algorithm has to decide when to buy, knowing only that we have gone skiing t times

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• If $t \ge 10$, we buy at the 10^{th} time, so cost is:

$$\frac{C_A}{C_{opt}} = \frac{100 \cdot 9 + 1000}{1000} = 1.9$$

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- Goal: maximize probability of dating the best person

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- If $\pi(k) = 1$, then $1 P_k$ is the probability that we picked a person between [t+1, k-1], which means someone in this range better than the first t people.

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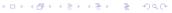
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- Optimizing we get that we should set t = n/e, which gives us 1/e probability.
- Wait a second, where is the competitive analysis?



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- That is, as we get older, we become more desperate to find someone and lower our expectations...

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 - If request is in cache, we have a $hit \leftrightarrow$ request takes negligible time

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- Simplification: assume we only have cache and main memory.

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Today, we will analyze the **Least Recently Used** heuristic. We will assume that *the size of our cache is k pages*.

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- **1** Least Recently Used (LRU): k-competitive
- **2 Random**: *k*-competitive
- **§** First-in, First-out (FIFO): *k*-competitive
- Least Frequently Used (LFU): NOT competitive



Theorem

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- Upper bound: divide input sequence into phases.
 - First phase starts immediately after our algorithm first faults, ends right after the algorithm faults *k* more times
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 - and so on...

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LRU Analysis - Example

Examples of phases, for k = 3:

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- Need to prove that OPT will fault at least once per phase.
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• If each page faulted once in a phase.

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- Proof: Look at last fault page in previous phase.

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- **Claim:** in the beginning of each phase, content of *OPT* and content of our algorithm *A* intersect in at least one page.
- Since *OPT* and *A* had a common page, then *OPT* must have faulted as well (since each page faulted in this phase)

Theorem

Any deterministic algorithm for paging with k pages is at least k-competitive!

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- **Input sequence:** at each step, request page that A doesn't have.

⁶Common lower bound technique for online algorithms, also commonly used online as well :)

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- A faults every single time.
- **Offline Algorithm:** on cache miss, delete page which is requested *furthest in the future*.
- When offline algorithm deletes a page, it's next delete happens after at least k steps.

Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
 - Stock Market
 - Dating
 - Skiing
 - Caching
 - Machine Learning (regret minimization)
 - many more...
- Competitive Analysis: measures performance of our algorithm against best algorithm that could see into the future

Acknowledgement

- Lecture based largely on:
 - Lecture 17 of Luca's Optimization class
 - Lectures 19 and 20 of Karger's 6.854 Fall 2004 algorithms course
 - [Motwani & Raghavan 2007, Chapter 13]
- See Luca's Lecture 17 notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture17.pdf
- See Karger's Lecture 19 notes at http://courses.csail.mit.edu/6.854/06/scribe/s22-online.pdf
- See Karger's Lecture 20 notes at http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf

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Motwani, Rajeev and Raghavan, Prabhakar (2007)

Randomized Algorithms