Lecture 14: Positive Semidefinite Matrices & Semidefinite Programming

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Overview

- Positive Semidefinite Matrices
- Why Semidefinite Programming?
- Convex Algebraic Geometry
- Application: Control Theory
- Conclusion
- Acknowledgements

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- Spectral theorem: any symmetric matrix in Mat(n, ℝ) has n real eigenvalues (counting with multiplicity), as well as an orthonormal basis (in ℝⁿ) for the eigenvectors.
- In other words, we can write

$$S = \sum_{i=1}^{n} \lambda_i u_i u_i^T$$

where $\lambda_i \in \mathbb{R}$ and $u_i \in \mathbb{R}^n$ such that $\langle u_i, u_j \rangle = \delta_{ij}$.

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- Practice problem: prove that these are all equivalent!

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minimize f(x)subject to $g_1(x) \ge 0$ \vdots $g_m(x) \ge 0$ $x \in \mathbb{R}^n$

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Constraints:

$$x_1 \cdot A_1 + \cdots + x_n \cdot A_n \succeq B$$

Minimize linear function $c^T x$

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minimize $c^T x$ subject to $x_1 \cdot A_1 + \dots + x_n \cdot A_n \succeq B$ $x \in \mathbb{R}^n$ $A_i, B \in \mathcal{S}^m(\mathbb{R})$ (fixed matrices)

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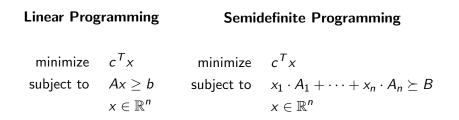
Where we use $C \succeq D$ to denote that $C - D \succeq 0$ (i.e., C - D is PSD).

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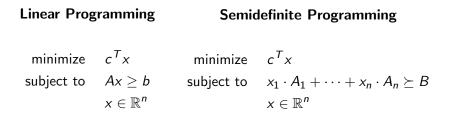
Linear Programming

 $\begin{array}{ll} \text{minimize} & c^{\mathsf{T}}x\\ \text{subject to} & Ax \geq b\\ & x \in \mathbb{R}^n \end{array}$

How does it generalize Linear Programming?



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Set A_i 's to be diagonal matrices, and $B = diag(b_1, \ldots, b_m)$

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 - equilibrium analysis of dynamics and control (flight controls, robotics, etc.)
 - robust optimization
 - statistics and ML
 - continuous games
 - software verification
 - filter design
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- See more here

https://windowsontheory.org/2016/08/27/

proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/

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• Is there a solution to the constraints at all?

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- How do we design *efficient algorithms* that find *optimal solutions* to Semidefinite Programs?

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Definition (Linear Matrix Inequalities)

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Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n A_i x_i \succeq B, A_i, B \in S^m \right\}$$

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Polyhedron:

Circle:

Hyperbola:

Elliptic curve:

Projected Spectrahedron

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

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A set $S \in \mathbb{R}^n$ is a *projected spectrahedron* if it has the form:

$$S = \left\{ x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^t \text{ s.t. } \sum_{i=1}^n A_i x_i + \sum_{j=1}^t B_j y_j \succeq C, A_i, B_j, C \in \mathcal{S}^m \right\}$$

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Example of Projected Spectrahedron

Projection of hyperbola:

Example of Projected Spectrahedron

Projection quadratic cone intersected with halfspace:

• To be able to optimize, we must be able to test whether a given point $x \in \mathbb{R}^n$ is inside our spectrahedron

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• Note that $x \in S$ is (by definition) equivalent to

$$Z=\sum_{i=1}^n A_i x_i - B \succeq 0$$

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- So, how do we efficiently check if $Z \succeq 0$?
- Symmetric Gaussian Elimination!
- We will use following characterizations of PSDness of symmetric $A \in \mathcal{S}^m$
 - all eigenvalues of A are *non-negative*
 - $A = LDL^T$ for some L lower triangular and unit diagonal, D diagonal and non-negative
 - **3** $z^T A z \ge 0$ for any $z \in \mathbb{R}^m$
 - 🔮 Any principal minor of A has non-negative determinant , 🚛 🖉

- Input: symmetric matrix $A \in S^m$
- **Output:** YES if $A \succeq 0$, NO otherwise (and output $z \in \mathbb{R}^m$ such that $z^T A z < 0$)

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• Our algorithm runs in time strongly polynomial.

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Setup:

• Linear difference equation

$$x(t+1) = Ax(t), \quad x(0) = x_0$$

• Discrete-time dynamical system.¹

¹When A non-negative and x_0 non-negative we have Markov chains $z \rightarrow z \rightarrow z \rightarrow 0$

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 - Temperatures of objects
 - Size of population
 - Voltage of electrical circuits
 - Concentration of chemical mixtures

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- System is stable iff $|\lambda_i(A)| < 1$

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SDP viewpoint:

• Lyapunov functions (generalize *energy* in systems). Functions on x(t) decrease monotonically on trajectories of the system.

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• To make these monotonically decreasing, we need:

$$egin{aligned} V(x(t+1)) &\leq V(x(t)) \Leftrightarrow x(t+1)^T P x(t+1) - x(t)^T P x(t) \leq 0 \ &\Leftrightarrow x(t)^T A^T P A x(t) - x(t)^T P x(t) \leq 0 \ &\Leftrightarrow A^T P A - P \preceq 0 \end{aligned}$$

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Theorem

Given matrix $A \in \mathbb{R}^{m \times m}$, the following conditions are equivalent:

- **1** All eigenvalues of A are inside unit circle, i.e. $|\lambda_i(A)| < 1$
- **2** There is $P \in S^m$ such that

$$P \succ 0, \quad A^T P A - P \prec 0$$

Setup:

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where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

If we properly choose control input u(t) we can make our system x(t) behave in a way that we want (say, to stabilize an unstable system)

Setup:

• Linear difference equation, with *control input*

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- If we properly choose control input u(t) we can make our system x(t) behave in a way that we want (say, to stabilize an unstable system)
- Want to do it by setting the control input to be u(t) = Kx(t) for some fixed K (so that we use the system as its own feedback)

Setup:

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where $A \in \mathbb{R}^{m \times m}$, $B \in \mathbb{R}^{m \times k}$

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 Wait, this ain't no SDP! But we can make it into SDP with some matrix manipulations.

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- Check out connections to Sum of Squares and a **bold**² attempt to have one algorithm to solve all problems! (i.e., one algorithm to rule them all)

https://windowsontheory.org/2016/08/27/

proofs-beliefs-and-algorithms-through-the-lens-of-sum-of-squares/

²pun intended

Acknowledgement

- Lecture based largely on:
 - [Blekherman, Parrilo, Thomas 2012, Chapter 2]

References I



Blekherman, Grigoriy and Parrilo, Pablo and Thomas, Rekha (2012) Convex Algebraic Geometry

