Lecture 13: Linear Programming Relaxation and Rounding

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

June 10, 2024

Overview

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- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

Example

Maximum Independent Set:

$$G(V, E) \text{ graph.}$$

$$\text{Independent set } S \subseteq V \text{ such that } u, v \in S \Rightarrow \{u, v\} \notin E.$$

Integer Linear Program:

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   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If solution has *fractional values*, then we have to devise *rounding procedure* that transforms

      fractional solutions $\rightarrow$ integral solutions

$$opt(LP) \leq \text{ rounded solution } \leq c \cdot opt(ILP)$$

Not all LPs created equal

When solving LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

it is important to understand *geometry of feasible set* & how nice the *corner points* are, as they are the candidates to *optimum* solution.

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- **Vertex Solutions:** a solution $x \in P$ is a vertex solution if $\nexists y \neq 0$ such that $x + y \in P$ *and* $x - y \in P$

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- **Basic Solutions:** let $supp(x) := \{i \in [n] \mid x_i > 0\}$ be the set of nonzero coordinates of x. Then $x \in P$ is a basic solution $\Leftrightarrow$ the columns of A indexed by $supp(x)$ are linearly independent.

- Part I
  - Why Relax & Round?
- Vertex Cover
- Set Cover
- Conclusion
- Acknowledgements

Vertex Cover

Setup:

- **Input:** a graph $G(V, E)$.
- **Output:** Minimum number of vertices that "touches" all edges of graph. That is, minimum set S such that for each edge $\{u, v\} \in E$ we have
$$|S \cap \{u, v\}| \geq 1.$$

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- If added elements to S k times, then $|S| = 2k$ and G has a matching of size k, which means that optimum vertex cover is at least k.
- Thus, we get a 2-approximation.

What can go wrong in the weighted case?

Original Algo

Heuristic: pick lowest weight only

Vertex Cover - LP relaxation

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4. Round LP as follows: round z_v to nearest integer.

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3. Round z_v to nearest integer. That is $y_v = \begin{cases} 1, & \text{if } z_v \geq 1/2 \\ 0, & \text{if } 0 \leq z_v < 1/2 \end{cases}$

Note that $y_v \leq 2z_v$

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6. Cost of y is:

$$\sum_{u \in V} c_u \cdot y_u \leq \sum_{u \in V} c_u \cdot (2 \cdot z_u) \leq 2 \cdot OPT(ILP)$$

Set Cover

Setup:

- **Input:** a finite set U and a collection $S_1, S_2, \ldots, S_n$ of subsets of U.
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2. Suppose we end up with fractional solution $z \in [0,1]^n$ when we solve the LP above. Now need to come up with a rounding scheme.
3. Can we just round each coordinate z_i to the nearest integer (like in vertex cover)?
4. Not really. Say $v \in U$ is in 20 sets, and we got $z_i = 1/20$ for each of the sets $v \in S_i$. Then rounding procedure above would not select any such set!

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4. Expected cost of the sets is $\sum_{i=1}^{n} w_i \cdot z_i$, which is the optimum for the LP. But will this process cover U?

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- As long as we select one of S_i's above we are good (w.r.t. v)
- We select S_i with probability z_i such that

$$\sum_{i=1}^{k} z_i \geq 1$$

Because z is a solution to our LP

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- As long as we select one of S_i's above we are good (w.r.t. v)
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  $$\sum_{i=1}^{k} z_i \geq 1$$

  Because z is a solution to our LP
- What is probability that v is covered in Random Pick?

Analyzing Random Pick

Let's consider the Random Pick process from point of view of $v \in U$.

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- By perseverance! :)

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Lemma (Probability of Covering an Element)

In a sequence of k independent experiments, in which the i^{th} experiment has success probability p_i, and

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- Thus probability of failure is

$$\prod_{i=1}^{k} (1 - p_i) \leq \prod_{i=1}^{k} e^{-p_i} = e^{-p_1 - \cdots - p_k} \leq 1/e$$

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4. Union bound, with probability ≤ 0.55 either run for more than t times, or our solution has weight $\geq 2\omega$
5. Thus, with probability ≥ 0.45 we stop at t iterations **and** construct solution to set cover with cost $\leq 2t \cdot OPT(ILP)$

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4. Solve LP optimally using efficient algorithm.
   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If have *fractional values*, *rounding procedure*

      Randomized Rounding algorithm, with probability ≥ 0.45 we get

$$cost(\text{rounded solution}) \leq 2 \cdot (\ln(|U|) + 3) \cdot OPT(ILP)$$

Conclusion

- Integer Linear programming - very general, and pervasive in (combinatorial) algorithmic life
- ILP NP-hard
- Rounding for the rescue!
- Solve LP and round the solution
    - Deterministic rounding when solutions are nice
    - Randomized rounding when things a bit more complicated

Acknowledgement

- Lecture based largely on:
  - Lectures 7-8 of Luca's Optimization class
- See Luca's vertex cover notes at `https://lucatrevisan.github.io/teaching/cs261-11/lecture07.pdf`
- See Luca's set cover notes at `https://lucatrevisan.github.io/teaching/cs261-11/lecture08.pdf`