Lecture 12: Applications of LP Duality

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Overview

- Game Theory - Minimax Theorems
- Learning Theory - Boosting
- Combinatorics - Bipartite Matching
- Conclusion
- Acknowledgements

Two-player games

Setup:

- Two players (Alice and Bob)
- Each player has a (finite) set of strategies $S_A = \{1, \ldots, m\}$ and $S_B = \{1, \ldots, n\}$

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Table: Battle of the sexes payoff matrices

Nash Equilibrium

Assuming players are rational, i.e. want to maximize their payoffs, we have:

Definition (Nash Equilibrium)

A strategy profile (i, j) is called a Nash equilibrium if the strategy played by each player is optimal, *given the strategy of the other player*. That is:

1. $A_{ij} \geq A_{kj}$ for all $k \in S_A$
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	Silent	Snitch
Silent	(-1,-1)	(-10,0)
Snitch	(0,-10)	(-5,-5)

Table: Prisoner's dilemma

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A mixed strategy is a probability distribution over a set of pure strategies S. If Alice's strategies are $S_A = \{1, \dots, n\}$, her mixed strategies are:

$$\Delta_A := \{x \in \mathbb{R}^n \mid x \geq 0 \text{ and } \|x\|_1 = 1\}$$

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$$v_A(x, y) = \sum_{(i,j) \in S_A \times S_B} A_{ij} x_i y_j = x^T A y$$

$$v_B(x, y) = \sum_{(i,j) \in S_A \times S_B} B_{ij} x_i y_j = x^T B y$$

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- One mixed Nash equilibrium: $x = y = (1/2, 1/2)$

Von Neumann's Minimax Theorem

Theorem

In a zero-sum game, for any payoff matrix $A \in \mathbb{R}^{m \times n}$:

$$\max_{x \in \Delta_A} \min_{y \in \Delta_B} x^T A y = \min_{y \in \Delta_B} \max_{x \in \Delta_A} x^T A y$$

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Left hand side can be written as

$$\begin{aligned} \max \quad & s \\ \text{s.t.} \quad & s \leq (x^T A)_j \quad \text{for } j \in S_B \\ & \sum_{i \in S_A} x_i = 1 \\ & x \geq 0 \end{aligned}$$

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For given $y \in \Delta_B$:

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Proof of Duality

- Game Theory - Minimax Theorems
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Learning Theory

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- *Weak learning assumption*:

  For any distribution $q \in \Delta_{\mathcal{X}}$, there is a hypothesis $h \in \mathcal{H}$ which is wrong less than half the time.

$$\exists \gamma > 0,\ \forall q \in \Delta_{\mathcal{X}},\ \exists h \in \mathcal{H}, \quad \Pr_{x \sim q}[h(x) \neq c(x)] \leq \frac{1-\gamma}{2}$$

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- Surprisingly, weak learning assumption implies something much stronger: it is possible to *combine* classifiers in $\mathcal{H}$ to construct a *classifier* that is *always right* (known as *strong learning*).

Boosting

Theorem

Let $\mathcal{H}$ be a set of hypotheses satisfying weak learning assumption. Then there is distribution $p \in \Delta_{\mathcal{H}}$ such that the weighed majority classifier

$$c_p(x) := \begin{cases} 1, & \textit{if } \sum_{h \in \mathcal{H}} p_h \cdot h(x) \geq 1/2 \\ 0, & \textit{otherwise} \end{cases}$$

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- In particular, right hand side implies weighted classifier given by optimum solution p^* *always* correct.

Proof of Correctness of Classifier

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- **Combinatorics - Bipartite Matching**
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- Breakthrough result of [Fenner, Gurjar and Thierauf 2019]
- We will see just a piece of the proof

Bipartite Matching & Circulation

- Given an even cycle $C = (e_1, e_2, \ldots, e_{2k})$, we say that the *circulation* of C is given by

$$circ(C) = |w(e_1) - w(e_2) + \ldots + w(e_{2k-1}) - w(e_{2k})|$$

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- Lemma: if we assign weights $w(e_i)$ such that $circ(C) \neq 0$ for each cycle C of the bipartite graph G, then we get that the minimum weight PM is unique!
- The approach of [Fenner, Gurjar and Thierauf 2019] is to construct a set of weights which make all circulations non-zero!
  - To do that, they iteratively construct a weight assignment that kills small cycles (i.e., make their circulation non-zero)
  - Once we have a bipartite graph with no cycles of length $2k$, then in next iteration we kill cycles of length up to $4k$
  - show that no cycles of length $2k \Rightarrow$ *few cycles* of length $4k$ – similar to Karger's min cut lemma!

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- Proof: LP duality! (complementary slackness)
  - Linear programs:

Primal

$$\begin{aligned} \min \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & x \geq 0 \\ & \sum_{e \in \delta(u)} x_e = 1 \\ & \text{for } u \in L \sqcup R \end{aligned}$$

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Dual

$$\begin{aligned} \max \quad & \sum_{u \in L \sqcup R} y_u \\ \text{s.t.} \quad & y_u + y_v \leq w_e \\ & \text{for } e = \{u, v\} \in E \end{aligned}$$

  - Complementary slackness says $x_e \neq 0$ in primal, where $e = \{u, v\}$ $\Rightarrow y_u + y_v = w_e$ in dual optimal.

Complementary Slackness & Circulation

Conclusion

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- General mathematical programming very hard (how hard do you think it is?)
- Special cases have very striking applications!

Today and last lecture: *Linear Programming*

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Today and last lecture: *Linear Programming*

- Linear Programming and Duality - fundamental concepts, lots of applications!
  - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
  - Applications in Game Theory (minimax theorem)
  - Applications in Learning Theory (boosting)
  - Applications in Parallel Computation/Derandomization (Perfect Matching)
  - many more

Acknowledgement

- Lecture based largely on:
  - Lectures 3-6 of Yarom Singer's Advanced Optimization class
  - [Schrijver 1986, Chapter 7]
  - Personal Communication with Rohit
- See Yarom's notes at `https://people.seas.harvard.edu/~yaron/AM221-S16/schedule.html`

References I

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