Lecture 11: Linear Programming and Duality Theorems

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Overview

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Conclusion
- Acknowledgements
- Proof of Fundamental Theorem of Linear Inequalities

Mathematical Programming

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- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(\mathbf{x}) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n + b = c^T x + b$$

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We can *always* represent LPs in *standard form*:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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- Other problems, such as *data fitting, linear classification* can be modelled as linear programs.

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   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
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4. How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs?

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Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$*, and* $t := \text{rank}\{a_1, \ldots, a_m, b\}$*. Then either*

1. b *is a non-negative linear combination of linearly independent vectors from* $a_1, \ldots, a_m$*, or*
2. *there exists a hyperplane* $H := \{x \mid c^T x = 0\}$ *s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
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Remark

Any hyperplane H with the above property is known as a *separating hyperplane*.

Farkas' Lemma

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *$y^T b \geq 0$ for each $y \in \mathbb{R}^m$ such that $y^T A \geq 0$*

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Equivalent formulation

Lemma (Farkas Lemma - variant 1)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$*

Farkas' Lemma

Equivalent formulation

Lemma (Farkas Lemma - variant 2)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

1. *There exists $x \in \mathbb{R}^n$ such that $Ax \leq b$*
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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$

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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$
- Now apply the original version of the lemma

Linear Programming Duality

Consider our linear program:

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\begin{aligned} y^T A \leq c^T &\Rightarrow y^T Ax \leq c^T x \\ &\Rightarrow y^T b \leq c^T x \end{aligned}$$

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$$\begin{aligned} y^T A \leq c^T & \Rightarrow y^T A x \leq c^T x \\ & \Rightarrow y^T b \leq c^T x \end{aligned}$$

- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

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Consider the following linear programs:

Primal LP

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minimize	$c^T x$	maximize	$y^T b$
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- From previous slide

$$y^T A \leq c^T \Rightarrow y^T b \text{ is a lower bound on value of Primal}$$

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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

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- When is the above inequality tight?

Strong Duality

Primal LP

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Theorem (Strong Duality)

If primal LP and dual LP are feasible, then

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1. Since we have proved weak duality, suffices to show that the following LP has a solution:

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & y^T A \leq c^T \\ & c^T x - y^T b \leq 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

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2. Apply variant 2 of Farkas' lemma on the system above.

Proof of Strong Duality

1. LP from previous page encoded by:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & A^T \\ -I & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ 0 \\ c \\ 0 \end{pmatrix}$$

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2. Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \quad v^T \quad \lambda \quad w^T) \geq 0$ such that $zB = 0$ then we have $u^Tb - v^Tb + w^Tc \geq 0$

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3. If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so

$$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$

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2. Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \quad v^T \quad \lambda \quad w^T) \geq 0$ such that $zB = 0$ then we have $u^T b - v^T b + w^T c \geq 0$
3. If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so

$$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$

4. If $\lambda = 0$, let x, y be feasible solutions (which we assumed to exist). Then $x \geq 0, Ax = b$ and $y^T A \leq c^T$. Thus

$$c^T w \geq y^T Aw = 0 \geq (v^T - u^T)Ax = (v^T - u^T)b$$

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \leq b$$

have at least one solution, and suppose that inequality

$$c^T x \leq \delta$$

holds whenever x satisfies $Ax \leq b$. Then, for some $\delta' \leq \delta$ the linear inequality

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Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Complementary Slackness

- If the optima in both primal and dual is finite, and x, y are feasible solutions, the following are equivalent:
  1. x, y are optimal solutions to the primal and dual
  2. $c^T x = y^T b$
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- 2 and 3 are equivalent as we can write

$$c^T x - y^T b = c^T x - y^T A x = (c^T - y^T A)x = \sum_{i=1}^{n} (c_i - y^T A_i) x_i$$

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Today: *Linear Programming*

- Linear Programming and Duality - fundamental concepts, lots of applications!
  - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
  - Applications in Game Theory (minimax theorem)
  - Applications in Learning Theory (boosting)
  - many more

Acknowledgement

- Lecture based largely on:
  - [Schrijver 1986, Chapter 7]

Proof of Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, and $t := \text{rank}\{a_1, \ldots, a_m, b\}$. Then either

1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or*
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
   - $c^T b < 0$
   - $c^T a_i \geq 0$
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- Since 1 and 2 mutually exclusive, choose linearly independent $\mathcal{L}_0 := \{a_{i_1}, \ldots, a_{i_n}\}$
- We will perform an iterative procedure:

Proof of Fundamental Theorem of Linear Inequalities

Iterative procedure, starting with $\mathcal{L}_0$:

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- To conclude the proof, need to show that this procedure always terminates. If process doesn't terminate, there are two times $r < t$ such that $\mathcal{L}_r = \mathcal{L}_t$
- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.

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- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.
- $\mathcal{L}_r = \mathcal{L}_t \Rightarrow a_\ell$ has also been added from some $\mathcal{L}_{k'}$ for some $r \leq k' < t$.

Proof of Fundamental Theorem of Linear Inequalities

- Say a_r was removed at iteration k and added back at iteration k' so $r \leq k < k' < t$
- Let c be the vector defining the hyperplane at the k' iteration (when we added a_r back to the set), and let $\mathcal{L}_k = \{a_{i_1}, \ldots, a_{i_n}\}$
- Now, above implies the following contradiction:

$$0 > c^T b = c^T(\lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} c^T a_{i_1} + \cdots \lambda_{i_n} c^T a_{i_n} \geq 0$$

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- Second inequality holds term by term:
    -

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