Lecture 10: Fundamental Theorem of Markov Chains, Page Rank

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Overview

- Main Tools
 - Linear Algebra Background
 - Perron-Frobenius
- Main Applications
 - Fundamental Theorem of Markov Chains
 - Page Rank
- Acknowledgements

• Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there is a vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

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- Example:

Lemma (Positivity Lemma)

If $A \in \mathbb{R}^{n \times n}$ is a positive matrix and $u, v \in \mathbb{R}^n$ are distinct vectors such that $u \ge v$, then Au > Av. Moreover, there exists $\varepsilon > 0$ such that $Au > (1 + \varepsilon)Av$.

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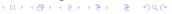
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• the moreover part just follows from taking small enough ε .



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Let $A \in \mathbb{R}^{n \times n}$ be a positive matrix (i.e., all its entries are positive). Then, the following hold:

- **1** $\rho(A)$ is an eigenvalue, and it has a positive eigenvector
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 - Let u be the vector defined by $u_i = |v_i|$. Then, we have

$$(Au)_i = \sum_j A_{ij}u_j \ge |\sum_j A_{ij}v_j| = |\lambda v_i| = \rho(A) \cdot u_i$$

so $Au \ge \rho(A)u$.



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By Gelfand's formula we would have

$$\rho(A) = \lim_{n \to \infty} \|A^n\|_F^{1/n} \ge (1 + \varepsilon)\rho(A)$$

which is a contradiction. So equality must hold.



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- If we had another eigenvalue $\lambda \neq \rho(A)$ in the circumference $|\mu| = \rho(A)$, where z is the eigenvector corresponding to λ , by the previous slide, we know that w defined as $w_i = |z_i|$ satisfies

$$Aw = \rho(A)w \Leftrightarrow \sum_{j} A_{ij}w_{j} = \rho(A) \cdot |z_{i}| = |\lambda z_{i}| = |\sum_{j} A_{ij}z_{j}|$$

for every $1 \le i \le n$

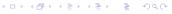
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• Lemma: if the conditions above hold, then there is $\alpha \in \mathbb{C}$ nonzero such that $\alpha z \geq 0$

Proof by squaring both sides and using complex conjugates.



• But if $\alpha z \geq 0$ and a nonzero vector, we have

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- $u \beta v \neq 0$ since the vectors are linearly independent
- But for each $1 \le i \le n$

$$\rho(A) \cdot (u - \beta v)_i = (A(u - \beta v))_i > 0$$

which contradicts our choice of β . Thus, there cannot be two linearly independent eigenvectors.

Perron-Frobenius

Theorem (Perron-Frobenius)

If a non-negative matrix $A \in \mathbb{R}^{n \times n}$ is aperiodic and irreducible, then the following hold:

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 - By previous lecture, we saw that A being aperiodic and irreducible implies that there is m > 0 such that A^m has all positive entries.
 - Apply Perron's theorem to A^m and note that the eigenvalues of A^m are λ_i^m , where λ_i are the eigenvalues of A

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Any finite, irreducible and aperiodic Markov Chain has the following properties:

- **1** There exists a unique stationary distribution π , where $\pi_i > 0$ for all $i \in [n]$
- **②** The sequence of distributions $\{p_t\}_{t\geq 0}$ will converge to π , no matter what the initial distribution is
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 The transition matrix P is non-negative, irreducible and aperiodic. So we can apply Perron-Frobenius and prove items 1 and 2.

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 - This eigenvector is $\pi!$
 - All random walks converge to π , as we wanted to show.

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- Intuition: if many other pages link to a particular page, then the linked page must be important!

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- This modification does not change "relative importance" of vertices

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 - Hannah Cairns notes on Perron-Frobenius (see link in course webpage)
 - Lap Chi's notes
 - [Motwani & Raghavan 2007, Chapter 6]
- See Lap Chi's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf
- Also see Lap Chi's notes
 https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf for a proof of fundamental theorem of Markov chains for undirected graphs.

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