#### Lecture 8: Sublinear Time Algorithms

Rafael Oliveira

University of Waterloo Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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### Overview

- Introduction
  - Why Sublinear Time Algorithms?
  - Warm-up Problem
- Main Problem
  - Number of Connected Components
- Acknowledgements

Sometimes big data does not come to us all at once (think streaming), but instead we *can query small pieces* of it.

Sometimes big data can also *change over time*, so we need a *robust* answer and/or be able to solve problem quickly multiple times.

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- Many more...

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Connects to randomized algorithms, approximation algorithms, parallel algorithms, complexity theory, statistics, learning

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What we *can* do:

- Can answer **for most** or **averages** or **approximate** type statements *with high probability* 
  - are most individuals connected via friendships?
  - are most individuals connected by at most 6 degrees of separation?
  - approximately how many people are left handed?
  - is my program correct on most inputs

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Randomized & Approximate algorithms.

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- Samples
  - get samples from certain distribution/input at each step

- Input: *m* points and a distance matrix *D* such that
  - $D_{ij} \leftarrow \text{distance from } i \text{ to } j$
  - D symmetric and satisfies triangle inequality

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- **Output:** Indices  $k, \ell$  such that

$$D_{k\ell} \geq D_{ab}/2$$

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#### 2-multiplicative algorithm

• Pick k arbitrarily

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Correctness

$$egin{aligned} D_{ab} &\leq D_{ak} + D_{kb} \ &\leq D_{k\ell} + D_{k\ell} = 2 \cdot D_{k\ell} \end{aligned}$$

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Is this the best we can do?

• Let D be following: distance matrix  $D_{i,i} = 0$ ,  $\forall i \in [m]$  and  $D_{i,j} = 1$  otherwise

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- Practice problem: prove that it would take Ω(N) time (i.e. number of queries) to decide if diameter is 1 or 2 - δ

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• Input: graph G(V, E) in *adjacency list* representation.  $\epsilon > 0$ .

$$n=|V|, m=|E|, N=m+n$$

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#### Lemma (# Connected Components)

Let G(V, E) be a graph. For vertex  $v \in V$ , let  $n_v \leftarrow \#$  vertices in connected component of v. Let C be number of connected components of G. Then:

$$C = \sum_{v \in V} \frac{1}{n_v}$$

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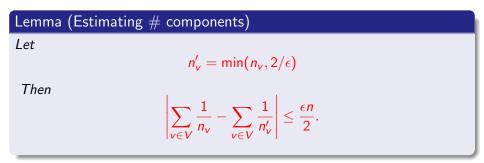
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# Lemma (Estimating # components) Let $n'_{v} = \min(n_{v}, 2/\epsilon)$ Then $\left|\sum_{v \in V} \frac{1}{n_{v}} - \sum_{v \in V} \frac{1}{n'_{v}}\right| \le \frac{\epsilon n}{2}.$

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How do we do this estimation?

Sample vertex v and run BFS starting at v, short-cutting if see  $2/\epsilon$  vertices.

# Connected Components - proof of lemma

#### Lemma (Estimating # components)

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Then

$$\left|\sum_{v\in V}\frac{1}{n_v}-\sum_{v\in V}\frac{1}{n'_v}\right|\leq \frac{\epsilon n}{2}.$$

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- Total running time  $O(1/\epsilon^4)$ .

To prove correctness we need to show that with probability  $\geq 3/4$  we have

$$\left|\frac{n}{s} \cdot \sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \sum_{v \in V} \frac{1}{n_v}\right| \le \epsilon n$$

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Dividing by n/s on both sides:

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By our previous lemma, and triangle inequality, enough to prove that w.h.p.

$$\left|\sum_{i=1}^{s} \frac{1}{n'_{v_i}} - \frac{s}{n} \cdot \sum_{v \in V} \frac{1}{n'_v}\right| \le \frac{\epsilon s}{2}$$

# Lemma and Triangle Inequality

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Then

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#### Theorem (Hoeffding's Inequality)

Let  $X_i$  be independent random variables, taking values in  $[a_i, b_i]$ ,  $X = \sum_{i=1}^{s} X_i$ . Then

$$\Pr[|X - \mathbb{E}[X]| \ge \ell] \le 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^{s} (b_i - a_i)^2}\right)$$

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Setting parameters of Hoeffing's theorem to our setting:

• 
$$a_i = 0, b_i = 1$$

• 
$$X_i = 1/n'_v$$
 with probability  $1/n$ 

(pick vertex uniformly at random)

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Since  $s = \Theta(1/\epsilon^2)$ , the result follows by choosing  $s = 8 \cdot (1/\epsilon^2)$ 

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# Acknowledgement

- Lecture based largely on Ronitt's notes.
- See Ronitt's notes at http://people.csail.mit.edu/ronitt/ COURSE/F20/Handouts/scribe1.pdf
- See also her notes for approximate MST http://people.csail. mit.edu/ronitt/COURSE/F20/Handouts/scribe2.pdf
- List of open problems in sublinear algorithms https://sublinear.info/index.php?title=Main\_Page