Lecture 07: Algebraic Techniques Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

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Overview

- Introduction
 - Why Algebraic Techniques in computer science?
 - Fingerprinting: String equality verification
- Main Problems
 - Polynomial Identity Testing
 - Randomized Matching Algorithms
 - Isolation Lemma
- Remarks
- Acknowledgements

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Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

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Suppose Alice and Bob each maintain the same large database of information.¹ They would like to check if their databases are *consistent*.

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Communication complexity setting, randomized algorithms, need to work with high probability.

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Fingerprinting mechanism:

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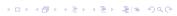


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- what happens when they are different?



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• Number of bits sent is $\tilde{O}(\log t + \log n)$. Choosing t = n solves it.

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- Can we check whether $P_1(x) \cdot P_2(x) = P_3(x)$ in O(n) operations?

Technique for string equality testing can be generalized to following setting:

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 Can amplify probability by running multiple times or by choosing larger set S.

Polynomial Identity Lemma

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Proof by induction in number of variables.

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- Output: does G have a perfect matching?

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• G has perfect matching $\Leftrightarrow \det(X)$ is a non-zero polynomial!²

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- Testing if G has a perfect matching is a special case of Polynomial Identity Testing!

²First proved by Edmonds.

- **Input:** bipartite graph G(L, R, E) with |L| = |R| = n
- Output: does G have a perfect matching?
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- We can compute the determinant efficiently in parallel

How? See the (future notes) in my CS 860 course.

Introduction

- Why Algebraic Techniques in computer science?
- Fingerprinting: String equality verification

Main Problems

- Polynomial Identity Testing
- Randomized Matching Algorithms
- Isolation Lemma
- Remarks
- Acknowledgements

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Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is *only one* set of minimum weight.

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- **2** α_v is *independent* of w(v), and w(v) chosen uniformly at random from [2n].
- **③** $Pr[v \text{ ambiguous}] \le 1/2n \Rightarrow_{\text{union bound}} Pr[∃ \text{ ambiguous element}] \le 1/2$

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- ① If two different sets A,B have minimum weight, then any element in $A\Delta B$ must be ambiguous.
- Probability that this happens is $\leq 1/2$. (step 8)



Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
 - Prof. Lau's notes
 - [Motwani & Raghavan 2007, Chapter 7]
 - [Korte & Vygen 2012, Chapter 10].
- See Prof. Lau's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf

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SIAM Journal on Computing

Svensson, Ola and Jakub Tarnawski (2017)
The matching problem in general graphs is in quasi-NC.

IEEE 58th Annual Symposium on Foundations of Computer Science

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d

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• Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_{\sigma}(V, F_{\sigma})$, where $F_{\sigma} = \{(i, \sigma(i))\}_{i=1}^{2n}$.

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- Each vertex in H_{σ} has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.

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- If σ only has even cycles, then H_{σ} gives us a perfect matching (by taking every other edge of the graph H_{σ} , ignoring orientation)

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• Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_{\sigma}(V, F_{\sigma})$, where $F_{\sigma} = \{(i, \sigma(i))\}_{i=1}^{2n}$.

• Otherwise, for each $\sigma \in S_{2n}$ (that has <u>odd cycle</u>), there is another permutation $r(\sigma) \in S_{2n}$ that is obtained by reversing odd cycle of H_{σ} containing vertex with <u>minimum index</u>.

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- If T_G has a matching, say, $\{1,2\}, \{3,4\}, \dots, \{2n-1,2n\}$, then take permutation $\sigma = (1\ 2)(3\ 4) \cdots (2n-1\ 2n)$

$$(-1)^{\sigma} \prod_{i=1}^{2n} [T_G]_{i,\sigma(i)} = (-1)^n \prod_{i=1}^n -x_{(2i-1)\sigma(2i-1)}^2 = \prod_{i=1}^n x_{(2i-1)\sigma(2i-1)}^2.$$