Lecture 6: Graph Sparsification

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Overview

- Introduction
  - Why Sparsify?
  - Warm-up Problem
- Main Problem
  - Graph Sparsification
- Acknowledgements

Why do we sparsify?

Often times graph algorithms for graphs $G(V, E)$ have runtimes which depend on $|E|$. If the graph is dense, i.e. $|E| = \omega(n^{1+\gamma})$, for $\gamma > 0$, then this may be *too slow*.

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- Settle for *approximate answers*
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- Applications in network connectivity

Graph Cuts

Definition (Graph Cut)

If $G(V, E, w)$ is a weighted graph, a *cut* is a partition of the vertices into two non-empty sets $V = S \sqcup \overline{S}$. The *value* of a cut is the quantity

$$w(S, \overline{S}) := \sum_{e \in E(S, \overline{S})} w_e.$$

Contraction of Edges

Definition (Edge Contraction)

Let $G(V, E)$ be a graph. If $e = \{u, v\} \in E$ is an edge of G, then the *contraction* of e is a new graph $H(V \cup \{z\} \setminus \{u, v\}, F)$ where we replace the vertices u, v by *one* vertex z, and any edge $\{u, x\} =: f \in E \setminus \{e\}$ is replaced by $\{z, x\} \in F$.

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- While there are more than 2 vertices in the graph:
  - Pick uniformly random edge and contract it
- Output the two subsets encoded by the two remaining vertices.

Analysis

Why does this work?

Intuition: picking a random edge uniformly at random “favours” *small cuts* (i.e. preserves them) with higher probability.

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Remark

The value of the minimum cut does note decrease after contraction.

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The probability that the algorithm outputs a minimum cut is at least $2/n(n-1)$, where $n = |V|$.

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  - Each vertex is a cut, so each vertex has degree $\geq c \Rightarrow$

$$\geq \frac{(n-i+1)\cdot c}{2} \quad \text{edges remain.}$$

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  - Contracting random edge, probability we kill cut $(S, \overline{S})$ is

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  - $\Pr[(S, \overline{S}) \text{ survives}] \geq (1 - 2/n) \cdot (1 - 3/n) \cdots (1 - 2/3) = 2/n(n-1)$

Hmmmmm, this is not with high probability...

- To improve success probability, repeat this randomized procedure t times (for which t?)
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- You will work on some running time improvements in your homework!

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This is all good, but we haven't "sparsified" anything so far!

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Let $G(V, E, w)$ be undirected weighted graph. For any cut $(S, \overline{S})$, let the weight of $(S, \overline{S})$ be

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We say that a graph $G(V, E)$ is *sparse* if $|E| = \tilde{O}(|V|)$.

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Question

How to make a graph sparse (nearly linear # edges) while approximating the value of every cut of a graph?

Graph Sparsification

- **Input:** graph $G(V, E, w_G)$, $\varepsilon > 0$.

$$n = |V|, \;\; m = |E|.$$

- **Output:** graph $H(V, F, w_H)$ such that *for every cut* $(S, \overline{S})$, we have

$$(1-\varepsilon) \cdot w_G(S, \overline{S}) \leq w_H(S, \overline{S}) \leq (1+\varepsilon) \cdot w_G(S, \overline{S})$$

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- *Assumption (for this class):* the input graph $G(V, E)$ is
  1. *unweighted*
  2. has *minimum cut value* $\Omega(\log n)$ — i.e., a large-ish cut

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Algorithm:

- Let $p \in (0, 1)$ be a parameter.
- For each edge $e \in E(G)$, with probability p, make e an edge of H with weight $w_H(e) = 1/p$.

Graph Sparsification

Idea:

- Set p to get correct expected value for both # edges in H and the value of each cut $(S, \overline{S})$ in H.

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Theorem ([Karger, 1993])

Let c be the value of the min-cut of G. Set

$$p = \frac{15 \ln n}{\varepsilon^2 \cdot c}.$$

Graph H given by algorithm from previous slide **approximates all cuts of** *G and has $O(p \cdot |E|)$ edges with probability $\geq 1 - 4/n$.*

Graph Sparsification

- Take a cut $(S, \overline{S})$. Suppose $k := w_G(S, \overline{S})$. Let
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  $$= |E(S, \overline{S})| = k = w_G(S, \overline{S})$$

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$$\Pr[|w_H(S, \overline{S}) - k| \geq \varepsilon \cdot k] \leq 2 \exp\left(-\frac{\varepsilon^2 kp}{3}\right) = 2n^{-5k/c}$$

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- So we can do a clever union bound!

Number of Cuts Lemma

Lemma (Number of small cuts)

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Practice problem: generalize our earlier proof on the # minimum cuts to this case.

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&= \sum_{\alpha = 1,2,4,8,\ldots} n^{-\alpha} \leq 4/n
\end{aligned}
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Another application of Chernoff gives us that H has the right number of edges $|F| \approx p \cdot |E|$ (i.e., sparse)

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- **Strong Connectivity:** a k-strong component is a maximal induced subgraph that is k-edge-connected. For each edge e, let s_e be the maximum value k such that there exists a k-strong component containing e.
- Sample edge e with probability $p_e = \Theta\left(\frac{\log n}{\varepsilon^2 \cdot s_e}\right)$ and weight $1/p_e$.

Acknowledgement

- Lecture based largely on Lap Chi's notes.
- See Lap Chi's Lecture 1 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L01.pdf`
- See Lap Chi's Lecture 3 notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L03.pdf`
- See Mohsen's notes for the general Benczur-Karger algorithm `https://people.inf.ethz.ch/gmohsen/AA18/Notes/S1.pdf`.

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