Lecture 3: Concentration Inequalities

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  - Markov's Inequality
- Higher Moments
  - Moments and Variance
  - Chebyshev's Inequality
  - Chernoff-Hoeffding's Inequality
- Acknowledgements

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When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

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Often times in algorithm analysis, running time is *concentrated* around expectation. This *concentration of measure* proves that our algorithms will *typically* run in time close to expectation.

Today's inequalities

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Today's inequalities II

Theorem (Chernoff-Hoeffding's Inequality)

Let $X_1, \ldots, X_n$ be independent indicator variables such that $\Pr[X_i = 1] = p_i$, where $0 < p_i < 1$. Let $X = \sum_{i=1}^{n} X_i$ and $\delta > 0$. Then

$$\Pr[X \geq (1+\delta) \cdot \mathbb{E}[X]] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}} \right]^{\mathbb{E}[X]},$$

and

$$\Pr[X \leq (1-\delta) \cdot \mathbb{E}[X]] \leq \exp\left(-\mathbb{E}[X] \cdot \delta^2/2 \right).$$

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Remark

Useful when we have no information beyond expected value (or when random variable difficult to analyze). Otherwise other inequalities much sharper!

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- Can it be modified to upper bound $\Pr[X \leq t]$?

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Definition (Covariance)

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X, Y] := \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])].$$

We say that X, Y are *positively correlated* if $\mathrm{Cov}[X, Y] > 0$ and *negatively correlated* if $\mathrm{Cov}[X, Y] < 0$.

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Proposition

- $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]$
- *If X, Y are independent, then* $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

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- Chebyshev:

$$\Pr[X \geq 3n/4] \leq \Pr[|X - n/2| \geq n/4] \leq \frac{n/4}{(n/4)^2} = 4/n$$

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Practice problem: Can you generalize Chebyshev's inequality to k^{th} order moments?

Sums of Independent Random Variables

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Central Limit Theorem: if we let $Z_n = \sum_{i=1}^{n} X_i$, where X_i independent copy of X, the random variable

$$Y_n = \frac{Z_n - n \cdot \mathbb{E}[X]}{\sqrt{n \cdot \sigma(X)^2}} \to \mathcal{N}(0, 1)$$

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

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- Not easy to work with, hard to generalize

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Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } t > 0.$$

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- The *moment generating function*

$$M_X(t) := \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{i \geq 0} \frac{t^i}{i!} \cdot X^i\right] = \sum_{i \geq 0} \frac{t^i}{i!} \cdot \mathbb{E}\left[X^i\right]$$

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- If $X = X_1 + X_2$, where X_1, X_2 are independent, note that

$$\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX_1} e^{tX_2}] = \mathbb{E}[e^{tX_1}] \cdot \mathbb{E}[e^{tX_2}]$$

Chernoff Bounds for Bounded Variables

Example (Heterogeneous Coin Flips)

Let $X_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{otherwise} \end{cases}$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$

1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$

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3. for $R \geq 6\mu$, $\Pr[X \geq R] \leq 2^{-R}$

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Theorem (Heterogeneous Coin Flips - lower tail)

1. $\Pr[X \leq (1-\delta) \cdot \mu] \leq \left[\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu}$
2. *if* $0 < \delta < 1$ *then* $\Pr[X \leq (1-\delta) \cdot \mu] \leq e^{-\mu\delta^2/2}$

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Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{n} X_i$. Then

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Proof uses *Hoeffding's lemma:* $\mathbb{E}[e^{t(X_i - \mathbb{E}[X_i])}] \leq \exp\left(\frac{t^2(b_i - a_i)^2}{8}\right)$

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected $\#$ heads is $n/2$. Chernoff-Hoeffding implies:

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$$\Pr[|\# \text{ heads } - n/2| \geq \sqrt{15 \cdot n}] \leq 2e^{-10}.$$

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- From previous slides:

$$\text{Markov: } \Pr[\#\text{ heads } \geq 3n/4] \leq 2/3$$
$$\text{Chebyshev: } \Pr[\#\text{ heads } \geq 3n/4] \leq 4/n.$$
$$\text{Chernoff: } \Pr[\#\text{ heads } \geq 3n/4] \leq e^{-n/24}.$$

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- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)
- Chernoff-Hoeffding bounds also hold for *negatively correlated* variables, because all we need is

$$\mathbb{E}[e^{t(X+Y)}] \leq \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$$

  This observation is very useful in many applications (also great source of final projects!)

Remarks

- It is often easier to compute moments by computing the moment generating functions
- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)
- Chernoff-Hoeffding bounds also hold for *negatively correlated* variables, because all we need is

$$\mathbb{E}[e^{t(X+Y)}] \leq \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$$

  This observation is very useful in many applications (also great source of final projects!)
- For instance: two edges appear in a random spanning tree is a negatively correlated event, thus Chernoff bounds are useful to analyze random spanning trees.

Acknowledgement

- Lecture based largely on Lap Chi's notes and [Motwani & Raghavan 2007, Chapters 3 and 4].
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L02.pdf`

References I

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