Lecture 1: Amortized Analysis

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Overview

- Introduction
 - Why amortized analysis?
 - Types of amortized analyses
- Examples of Data Structures Using Amortized Analysis
 - Aggregate Analysis
 - Accounting Method
 - Potential Method
- Acknowledgements

Why Amortized Analysis?

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Worst or average-case complexity of data structures

Data Structure	search	insertion	deletion
Doubly-Linked List	O(n)	O(1)	O(n)
Ordered Array	$O(\log n)$	O(n)	O(n)
Hash Tables ^a	O(1)	O(1)	O(1)
Balanced Binary Search Trees ^b	$O(\log n)$	$O(\log n)$	$O(\log n)$

^aAverage-case, although worst-case search time is $\Theta(n)$

^bAlso average-case. Worst-case complexity is O(height) of the tree, which can be $\Theta(n)$.

Why Amortized Analysis?

In **amortized analysis**, one averages the *total time* required to perform a sequence of data-structure operations over *all operations performed*.

Upshot of amortized analysis: worst-case cost *per query* may be high for one particular query, so long as overall average cost per query is small in the end!

Remark

Amortized analysis is a *worst-case* analysis. That is, it measures the average performance of each operation in the worst case.

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- Optential Method: one comes up with potential energy of a data structure, which maps each state of entire data-structure to a real number (its "potential"). Differs from accounting method because we assign credit to the data structure as a whole, instead of assigning credit to each operation.

One simple problem - several analyses

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- Putting it all together, we get:

$$\sum_{k=0}^{\lceil \log n \rceil} \lfloor n/2^k \rfloor < \sum_{k \ge 0} n/2^k = 2n$$

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- If we manage to do the above, then

Total cost \leq Total charged

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• Total cost \leq Total Charged $= 2 \times n$

Example of the accounting method

Formal Analysis of the accounting method

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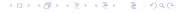
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• So if $\Phi_k - \Phi_0 \ge 0$ for all $k \ge 0$ (valid potential function) the total amortized cost is an upper bound on total cost.



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• Since each increment *only changes 1 bit from 0 to 1* each amortized cost is 2.

Example of the potential method

Discussion of the potential method

Acknowledgements

- Lecture largely based on Jeff Erickson's notes (with exercises!)
 http://jeffe.cs.illinois.edu/teaching/algorithms/notes/ 09-amortize.pdf
- More exercises and another example using all methods can also be found at the [CLRS] book, chapter 17. (see useful resources page)