Lecture 1: Amortized Analysis

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Overview

- Introduction
  - Why amortized analysis?
  - Types of amortized analyses
- Examples of Data Structures Using Amortized Analysis
  - Aggregate Analysis
  - Accounting Method
  - Potential Method
- Acknowledgements

Why Amortized Analysis?

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Worst or average-case complexity of data structures

Data Structure	search	insertion	deletion
Doubly-Linked List	$O(n)$	$O(1)$	$O(n)$
Ordered Array	$O(\log n)$	$O(n)$	$O(n)$
Hash Tables[a]	$O(1)$	$O(1)$	$O(1)$
Balanced Binary Search Trees[b]	$O(\log n)$	$O(\log n)$	$O(\log n)$

[a] Average-case, although worst-case search time is $\Theta(n)$

[b] Also average-case. Worst-case complexity is $O(\mathit{height})$ of the tree, which can be $\Theta(n)$.

Why Amortized Analysis?

In **amortized analysis**, one averages the *total time* required to perform a sequence of data-structure operations over *all operations performed*.

Upshot of amortized analysis: worst-case cost *per query* may be high for one particular query, so long as overall average cost per query is small in the end!

Remark

Amortized analysis is a *worst-case* analysis. That is, it measures the average performance of each operation in the worst case.

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3. **Potential Method:** one comes up with *potential energy* of a data structure, which maps each state of entire data-structure to a real number (its "potential"). Differs from accounting method because we assign credit to the data structure as a whole, instead of assigning credit to each operation.

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- Putting it all together, we get:

$$\sum_{k=0}^{\lceil \log n \rceil} \lfloor n/2^k \rfloor < \sum_{k \geq 0} n/2^k = 2n$$

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- If we manage to do the above, then

$$\text{Total cost} \leq \text{Total charged}$$

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- Total cost $\leq$ Total Charged $= 2 \times n$

Example of the accounting method

Formal Analysis of the accounting method

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- So if $\Phi_k - \Phi_0 \geq 0$ for all $k \geq 0$ (*valid potential function*) the total amortized cost is an *upper bound* on total cost.

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- Since each increment *only changes 1 bit from 0 to 1* each amortized cost is 2.

Example of the potential method

Discussion of the potential method

Acknowledgements

- Lecture largely based on Jeff Erickson's notes (with exercises!)

  `http://jeffe.cs.illinois.edu/teaching/algorithms/notes/09-amortize.pdf`

- More exercises and another example using all methods can also be found at the [CLRS] book, chapter 17. (see useful resources page)