Final Research Project for CS 466/666

your name * your other name †

Abstract

Write a summary of your main result/object of study.

1 Introduction

EXAMPLE INTRODUCTION.

Let $\mathbf{x} = (x_1, \dots, x_n)$ be a vector of variables $x_1, \dots, x_n$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n$ be a vector of elements $a_1, \dots, a_n$ from $\mathbb{R}$. A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ is hyperbolic with respect to a direction $\mathbf{e} := (e_1, \dots, e_n) \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$ and for all vectors $\mathbf{a} \in \mathbb{R}^n$, the univariate polynomial $f(t) := h(t\mathbf{e} - \mathbf{a})$ only has real zeros. By a result due to Gårding [Går59], each hyperbolic polynomial $h(\mathbf{x})$ is associated with its *hyperbolicity cone*, a closed convex cone denoted by $\Lambda_+(h, \mathbf{e})$ and defined as

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{a} \in \mathbb{R}^n \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative}\}.$$

Gårding also showed [Går59] that $\Lambda_+(h, \mathbf{e})$ can be equivalently defined as the closure of the connected component of $\{\mathbf{a} \in \mathbb{R}^n \mid h(\mathbf{a}) \neq 0\}$ that contains $\mathbf{e}$.

Hyperbolic polynomials and hyperbolicity cones originated in the theory of PDE in the works of Petrovsky and Gårding, and are of importance in combinatorics and optimization. Hyperbolicity cones are important objects in optimization, as they generalize semidefinite cones and Güler [Gül97] showed that one could generalize interior point methods of optimization to hyperbolicity cones. Since then the theory of hyperbolic programming has been vastly expanded, see [Ren04] and references therein.

Despite much progress on the optimization side of hyperbolic programming, the geometric and complexity theoretic aspects of hyperbolicity cones are much less understood.

On the geometric side, an important open question is concerned with how general these hyperbolicity cones are. *Spectrahedral cones*, that is, linear sections of the cone of positive semidefinite matrices, form the most well-known examples of hyperbolicity cones. The generalized Lax conjecture states that every hyperbolicity cone is also a spectrahedral cone, whereas the projected Lax conjecture states that every hyperbolicity cone is a linear projection of a spectrahedral cone. Despite much recent work and some impressive progress on these conjectures [NS15, Kum17], they remain open.

On the complexity theoretic side, very little is known about the complexity of representing hyperbolicity cones which are known to be spectrahedral. In the recent work [RRSW19], the authors prove exponential lower bounds even for approximate spectrahedral representations of

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non-explicit hyperbolicity cones which are spectrahedral. However, prior to the present work, no superpolynomial lower bound on the spectrahedral representation for an *explicit* hyperbolicity cone which is also spectrahedral was known. In the next section we present our main result and the overview of its proof, which is given formally in the next sections.

1.1 Main result and proof overview

State the main result that you are studying. We prove the following theorem:

Theorem 1.1. *Statement of the main result.*

High-level ideas of the proof: The high level idea guiding the proof of Theorem 1.1 comes from the steps:

1.
2.
3.

Explain a bit more the proof strategy above.

1.2 Related Work

Describe related work to the one that you researched.

1.3 Organization

In Section 4 we conclude and present some open problems.

2 Preliminaries

In this section, we establish the notation which will be used throughout the paper and some important background which we shall need to prove our claims in the next sections.

2.1 General Facts and Notations

We will work over the field $\mathbb{R}$ of real numbers. (for instance)

3 Technical Section

Present the main technical component here.

4 Conclusion and Open Problems

Conclude your survey/research paper and present some open questions, if there are any.

Acknowledgments

Acknowledge any discussions with other members of the class (if you discussed about this with anyone else and you found that it was helpful).

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