PSPACE Completeness

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Definition: a quantified boolean formula is a boolean formula where certain variables are quantified.
For instance $\phi = \exists x \forall y \left[(x \vee y) \wedge (\bar{x} \vee \bar{y}) \right]$

A fully quantified boolean formula is a quantified boolean formula where <u>every</u> variable is quantified.

$$TQBF := \{ \langle \phi \rangle \mid \phi \text{ is a true fully quantified boolean formula} \}$$

Theorem: TQBF is PSPACE-complete

Proof: we have already discussed that $TQBF \in PSPACE$.
Hence all we need to do is to show that given $A \in PSPACE$, then $A \leq_m TQBF$.
$A \in PSPACE \Rightarrow \exists\, c \in \mathbb{N}$ and TM M with $s_M(n) = O(n^c)$
s.t. $x \in A \Leftrightarrow M(x) = 1$.

Discussion:
Since we are trying to map $x \in A$ to a (true, fully quantified) boolean formula, we could try to generalize Cook-Levin theorem for our setting.
While M using space $O(n^c)$ implies that the width of the tableau is polynomially bounded, note that M can run for time $2^{O(n^c)}$ thus the height of the tableau is too large.
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tableau is too long.

The idea is to use quantifiers, together with Cook-Levin and the algorithm from Savitch's theorem, to construct a small fully quantified boolean formula.

Let us try and solve the more general problem of testing whether given two configurations c_1, c_2 in $G_{M,x}$ and a length parameter t.

We will denote by $\phi_{c_1,c_2,t}$ the quantified boolean formula that will take value 1 iff there is a $c_1 \to c_2$ path of length $\leq t$ in $G_{M,x}$.

Note that we want to construct $\phi_{c_{start}, c_{accept}, 2^{O(n^k)}}$.

Recall that we can describe a configuration of $M(x)$ with $O(n^k)$ bits, and similarly to the proof of Cook-Levin, we can express whether two configurations are equal, or whether $(c_1, c_2) \in E(G_{M,x})$ via a boolean formula $\phi_{c_1,c_2,1}$ of size $O(n^k)$.

What do we do when $t > 1$? (note that we can assume t is a power of 2)

Attempt #1: just as in Savitch's theorem, we can do

$$\phi_{c_1,c_2,t} = \exists m_1 \left[\phi_{c_1,m_1,t/2} \wedge \phi_{m_1,c_2,t/2} \right]$$

where m_1 corresponds to a configuration of M.

(thus m_1 corresponds to $O(n^k)$ bits.)

While $\phi_{c_1,c_2,t}$ computes the right value, its size is going to be $\Theta(tn^k)$ which will be too large when $t = 2^{O(n^k)}$.

Attempt #2: to reduce the size, let us use the universal quantifier to avoid the duplication of the inner formula! (Just as Savitch reused the space)

Let

$$\phi_{c_1,c_2,t} = \exists m_1 \forall c_3, c_4 \left[\Big((c_3,c_4) = (c_1,m_1) \vee (c_3,c_4) = (m_1,c_2) \Big) \Rightarrow \phi_{c_3,c_4,t/2} \right]$$

Now, the recursion for the quantified formula size of $\phi_{c_1,c_2,t}$

is $$S(t) \le S(t/2) + \underbrace{\gamma n^k}_{\text{new quantifiers and equality tests}}$$

$$\Rightarrow S(t) = O(n^k \cdot \log t)$$

$$\Rightarrow S(2^{O(n^k)}) = O(n^{2k})$$ and we are done. $\square$