Lecture 22: Zero-Knowledge Proofs

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Overview

- Why Zero Knowledge?
- Zero-Knowledge Proofs
- Conclusion
- Acknowledgements

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  - Can Alice convince Bob that she gave right file without giving any more *knowledge* beyond that she gave right file?

Zero-Knowledge Proofs

Proofs in which the verifier gains *no knowledge* beyond the validity of the assertion.

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- In both cases Alice conveyed *information*!

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- Verifier *does not trust* prover. Otherwise no need to verify proof!

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- In this setting, verifier *learns the isomorphism* (i.e., the proof)!

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- In the end, we will see a (zero-knowledge) proof for graph isomorphism as follows:

  Alice: I will not give you an isomorphism, but I will prove that I could give you one, if I wanted to.

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- How can we model the fact that verifier does not gain knowledge?!

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- Simulation $\Rightarrow$ V gained no new information!

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Definition (Perfect Zero Knowledge)

A proof system (P, V) is *perfect zero-knowledge* for language L if for every polynomial time, randomized verifier V^*, there is a randomized algorithm M^* such that for every $x \in L$ the following random variables are identically distributed:

- $\langle P, V^* \rangle(x)$, that is, output of interaction between prover P and verifier V^* on input x
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- The above captures the idea that V^* is not gaining any extra computational power by interacting with P, since same output could have been generated by M^*

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- Previous definition is a bit too strict to be useful, so we relax it.[1]
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1. With probability $\leq 1/2$, $M^*(x) = \perp$
2. Conditioned on $M^*(x) \neq \perp$, the following variables are identially distributed:
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- Note that whenever we don't fail, we output same distribution as the original protocol!

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- Has applications in
  - Modern cryptography
  - Credit Cards
  - Passwords
  - Complexity Theory (can use zero-knowledge to construct complexity classes)
  - Used in cryptocurrencies (validate transactions without giving details about transactions)

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  - Berkeley & MIT's 6.875 Lecture 14

    `https://inst.eecs.berkeley.edu/~cs276/fa20/slides/lec14.pdf`