Lecture 16: Semidefinite Programming Relaxation and MAX-CUT

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Overview

- Max-Cut SDP Relaxation & Rounding
- Conclusion
- Acknowledgements

Relax... & Round!

In our quest to get efficient (exact or approximate) algorithms for problems of interest, the following strategy is very useful:

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   1. If solution to SDP is *integral* and *one-dimensional*, then it is a solution to QP and we are done
   2. If solution has *higher dimension*, then we have to devise *rounding procedure* that transforms

      high dimensional solutions $\rightarrow$ integral & 1D solutions

      rounded SDP solution value $\geq c \cdot OPT(QP)$

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Max-Cut

Maximum Cut (Max-Cut):

$G(V, E)$ graph.

Cut $S \subseteq V$ and size of cut is

$$|E(S, \overline{S})| = |\{(u, v) \in E \ \mid \ u \in S, v \notin S\}|.$$

Goal: find cut of maximum size.

Example - Weighted Variant

Maximum Cut (Max-Cut):

$G(V, E, w)$ weighted graph. $\sum_{e \in E} w_e = 1$

Cut $S \subseteq V$ and weight of cut is the sum of weights of edges crossing cut.
Goal: find cut of maximum weight.

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Quadratic Program:

$$\begin{aligned} \text{maximize} \quad & \sum_{\{u,v\} \in E} \frac{1}{2} \cdot w_{u,v} \cdot (1 - x_u x_v) \\ \text{subject to} \quad & x_v^2 = 1 \quad \text{for } v \in V \end{aligned}$$

SDP Relaxation [Delorme, Poljak 1993]

$G(V, E, w)$ weighted graph, $|V| = n$ and $\sum_{e \in E} w_e = 1$

Semidefinite Program:

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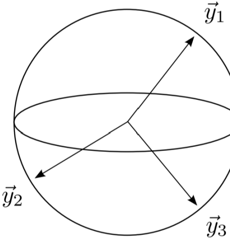

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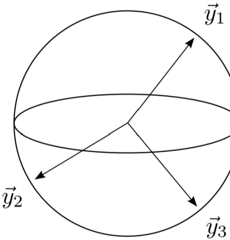

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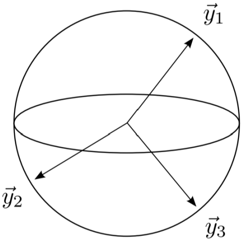

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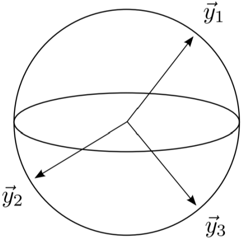

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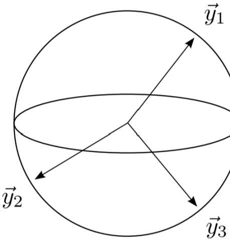

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- Geometrically, want vertices from our max-cut S to be as far away from the complement $\overline{S}$ as possible
- If all y_v's are in a one-dimensional space, then we get original quadratic program

$$OPT(SDP) \geq \text{Weight of Maximum Cut}$$

Example

Let's consider $G = K_3$ with equal weights on edges.

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- **Practice problem:** try this with C_5.

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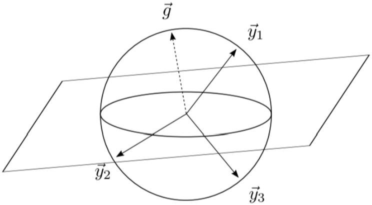

Figure 10.2: Vectors being separated by a hyperplane with normal $\vec{g}$.

Facts we need

- We can pick a random hyperplane through origin in polynomial time.

  sample vector $g = (g_1, \ldots, g_n)$ by drawing $g_i \in \mathcal{N}(0, 1)$

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  sample vector $g = (g_1, \ldots, g_n)$ by drawing $g_i \in \mathcal{N}(0, 1)$
- If g' is the projection of g onto a two dimensional plane, then $g'/\|g'\|_2$ is *uniformly distributed* over the unit circle in this plane.

Analysis of Rounding

- Probability that edge $\{u, v\}$ crosses the cut is same as probability that z_u, z_v fall in different sides of hyperplane

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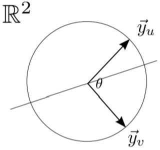

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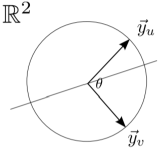

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- Probability of splitting z_u, z_v:

$$\Pr[\{u, v\} \text{ crosses cut}] = \frac{\theta}{\pi} = \frac{\cos^{-1}(z_u^T z_v)}{\pi} = \frac{\cos^{-1}(\gamma_{uv})}{\pi}$$

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- For $x \in [-1, 1]$, we have

$$\frac{\cos^{-1}(x)}{\pi} \geq 0.878 \cdot \frac{1 - x}{2}$$

  proof by elementary calculus.

Conclusion of rounding algorithm

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    3. With constant probability, our solution will be $\geq 0.878 OPT(\text{Max-Cut})$

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All of these are amazing final project topics!

Conclusion

- Mathematical programming - very general, and pervasive in (combinatorial) algorithmic life
- Mathematical Programming hard in general
- Sometimes can get SDP rounding!
- Solve SDP and round the solution
  - Deterministic rounding when solutions are nice
  - Randomized rounding when things a bit more complicated

Acknowledgement

- Lecture based largely on:
  - Lecture 14 of Anupam Gupta and Ryan O'Donnell's Optimization class

    `https://www.cs.cmu.edu/~anupamg/adv-approx/`
  - Chapter 6 of book [Williamson, Shmoys 2010]
- See their notes at

  `https://www.cs.cmu.edu/~anupamg/adv-approx/lecture14.pdf`

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