Lecture 13: Linear Programming Relaxation and Rounding

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Overview

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- Disadvantage of ILPs: capture even NP-hard problems (thus NP-hard)
- But we know how to solve LPs. Can we get partial credit in life?

Example

Maximum Independent Set:

$$G(V, E) \text{ graph.}$$

$$\text{Independent set } S \subseteq V \text{ such that } u, v \in S \Rightarrow \{u, v\} \notin E.$$

Integer Linear Program:

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & x_u + x_v \leq 1 \quad \text{for } \{u, v\} \in E \\ & x_v \in \{0, 1\} \quad \text{for } v \in V \end{aligned}$$

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   1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
   2. If solution has *fractional values*, then we have to devise *rounding procedure* that transforms

      fractional solutions → integral solutions

$$opt(LP) \leq \text{ rounded solution } \leq c \cdot opt(ILP)$$

Not all LPs created equal

When solving LP

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

it is important to understand *geometry of feasible set* & how nice the *corner points* are, as they are the candidates to *optimum* solution.

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- **Basic Solutions:** let $supp(x) := \{i \in [n] \mid x_i > 0\}$ be the set of nonzero coordinates of x. Then $x \in P$ is a basic solution $\Leftrightarrow$ the columns of A indexed by $supp(x)$ are linearly independent.

- Part I
    - Why Relax & Round?
- Vertex Cover
- Set Cover
- Conclusion
- Acknowledgements

Vertex Cover

Setup:

- **Input:** a graph $G(V, E)$.
- **Output:** Minimum number of vertices that "touches" all edges of graph. That is, minimum set S such that for each edge $\{u, v\} \in E$ we have
$$|S \cap \{u, v\}| \geq 1.$$

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- If added elements to S k times, then $|S| = 2k$ and G has a matching of size k, which means that optimum vertex cover is at least k.
- Thus, we get a 2-approximation.

What can go wrong in the weighted case?

Original Algo

Heuristic: pick lowest weight only

Vertex Cover - LP relaxation

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4. Round LP as follows: round z_v to nearest integer.

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6. Cost of y is:

$$\sum_{u \in V} c_u \cdot y_u \leq \sum_{u \in V} c_u \cdot (2 \cdot z_u) \leq 2 \cdot OPT(ILP)$$

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- **Set Cover**
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Set Cover

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3. Can we just round each coordinate z_i to the nearest integer (like in vertex cover)?
4. Not really. Say $v \in U$ is in 20 sets, and we got $z_i = 1/20$ for each of the sets $v \in S_i$. Then rounding procedure above would not select any such set!

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4. Expected cost of the sets is $\sum_{i=1}^{n} w_i \cdot z_i$, which is the optimum for the LP. But will this process cover U?

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- $v \in S_1, \ldots, S_k$ (for simplicity)
- As long as we select one of S_i's above we are good (w.r.t. v)
- We select S_i with probability z_i such that

$$\sum_{i=1}^{k} z_i \geq 1$$

  Because z is a solution to our LP

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Let's consider the Random Pick process from point of view of $v \in U$.

- $v \in S_1, \dots, S_k$ (for simplicity)
- As long as we select one of S_i's above we are good (w.r.t. v)
- We select S_i with probability z_i such that

$$\sum_{i=1}^{k} z_i \geq 1$$

  Because z is a solution to our LP
- What is probability that v is covered in Random Pick?

Analyzing Random Pick

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- By perseverance! :)

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Lemma (Probability of Covering an Element)

In a sequence of k independent experiments, in which the i^{th} experiment has success probability p_i, and

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- Thus probability of failure is

$$\prod_{i=1}^{k} (1 - p_i) \leq \prod_{i=1}^{k} e^{-p_i} = e^{-p_1 - \cdots - p_k} \leq 1/e$$

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4. Union bound, with probability ≤ 0.55 either run for more than t times, or our solution has weight $\geq 2\omega$
5. Thus, with probability ≥ 0.45 we stop at t iterations **and** construct solution to set cover with cost $\leq 2t \cdot OPT(ILP)$

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4. Solve LP optimally using efficient algorithm.
    1. If solution to LP has *integral values*, then it is a solution to ILP and we are done
    2. If have *fractional values*, *rounding procedure*

        Randomized Rounding algorithm, with probability ≥ 0.45 we get

$$cost(\text{rounded solution}) \leq 2 \cdot (\ln(|U|) + 3) \cdot OPT(ILP)$$

Conclusion

- Integer Linear programming - very general, and pervasive in (combinatorial) algorithmic life
- ILP NP-hard
- Rounding for the rescue!
- Solve LP and round the solution
    - Deterministic rounding when solutions are nice
    - Randomized rounding when things a bit more complicated

Acknowledgement

- Lecture based largely on:
  - Lectures 7-8 of Luca's Optimization class
- See Luca's vertex cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture07.pdf
- See Luca's set cover notes at https://lucatrevisan.github.io/teaching/cs261-11/lecture08.pdf