Lecture 11: Linear Programming and Duality Theorems

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Overview

- Part I
  - Why Linear Programming?
  - Structural Results on Linear Programming
  - Duality Theory
- Conclusion
- Acknowledgements
- Proof of Fundamental Theorem of Linear Inequalities

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- Special case is when all functions $f, g_1, \ldots, g_m$ are *linear* functions (called *Linear Programming* - LP for short)
- Traces of idea of LP in works of Fourier [Fourier 1823, Fourier 1824]
- Formally studied & importance of LP recognized in 1940's by Dantzig, Kantorovich, Koopmans and von Neumann.

What is a Linear Program?

A linear function $f : \mathbb{R}^n \to \mathbb{R}$ is given by

$$f(\mathrm{x}) = c_1 \cdot x_1 + \ldots + c_n \cdot x_n + b = c^T x + b$$

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We can *always* represent LPs in *standard form*:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

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- Other problems, such as *data fitting, linear classification* can be modelled as linear programs.

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   - How can we know that we found a minimum solution?
   - Do these solutions have nice description?
   - Do the solutions have *small bit complexity*?

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    - Do the solutions have *small bit complexity*?
4. How do we design *efficient algorithms* that find *optimal solutions* to Linear Programs?

Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$, *and* $t := \text{rank}\{a_1, \ldots, a_m, b\}$. *Then either*

1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or*
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
    - $c^T b < 0$
    - $c^T a_i \geq 0$
    - *H contains $t - 1$ linearly independent vectors from $a_1, \ldots, a_m$*

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Remark

The hyperplane H above is known as the *separating hyperplane*.

Farkas' Lemma

Lemma (Farkas Lemma)

Let $A \in \mathbb{R}^{m \times n}$ *and* $b \in \mathbb{R}^m$. *The following are equivalent:*

1. *There exists* $x \in \mathbb{R}^n$ *such that* $x \geq 0$ *and* $Ax = b$
2. $y^T b \geq 0$ *for each* $y \in \mathbb{R}^m$ *such that* $y^T A \geq 0$

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Equivalent formulation

Lemma (Farkas Lemma - variant 1)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Then exactly one of the following statements hold:

1. *There exists $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Ax = b$*
2. *There exists $y \in \mathbb{R}^m$ such that $y^T b > 0$ and $y^T A \leq 0$*

Farkas' Lemma

Equivalent formulation

Lemma (Farkas Lemma - variant 2)

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. The following are equivalent:

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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$

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- Let $M = [I \quad A \quad -A]$. Then $Ax \leq b$ has a solution iff $Mz = b$ has a non-negative solution $z \geq 0$
- Now apply the original version of the lemma

Linear Programming Duality

Consider our linear program:

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- If we look at what happens when we multiply $y^T A$, note the following:

$$\begin{aligned} y^T A \leq c^T &\Rightarrow y^T A x \leq c^T x \\ &\Rightarrow y^T b \leq c^T x \end{aligned}$$

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- Thus, if $y^T A \leq c^T$, then we have that $y^T b$ is a *lower bound* on the solution to our linear program!

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Consider the following linear programs:

Primal LP		*Dual LP*	
minimize	$c^T x$	maximize	$y^T b$
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Theorem (Weak Duality)

Let x be a feasible solution of the primal LP and y be a feasible solution of the dual LP. Then

$$y^T b \leq c^T x.$$

Remarks on Duality

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- When is the above inequality tight?

Strong Duality

Primal LP	
minimize	$c^T x$
subject to	$Ax = b$
	$x \geq 0$

Dual LP	
maximize	$y^T b$
subject to	$y^T A \leq c^T$

- let $\alpha^*, \beta^* \in \mathbb{R}$ be optimal values for primal and dual, respectively.

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Theorem (Strong Duality)

If primal LP and dual LP are feasible, then

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1. Since we have proved weak duality, suffices to show that the following LP has a solution:

$$\begin{aligned} \text{maximize} \quad & 0 \\ \text{subject to} \quad & y^T A \leq c^T \\ & c^T x - y^T b \leq 0 \\ & Ax = b \\ & x \geq 0 \end{aligned}$$

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2. Apply variant 2 of Farkas' lemma on the system above.

Proof of Strong Duality

1. LP from previous page encoded by:

$$B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A & 0 \\ -A & 0 \\ c^T & -b^T \\ 0 & A^T \\ -I & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b \\ -b \\ 0 \\ c \\ 0 \end{pmatrix}$$

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2. Variant 2 of Farkas' lemma gives that the system has solution iff for each $z = (u^T \quad v^T \quad \lambda \quad w^T) \geq 0$ such that $zB = 0$ then we have $u^T b - v^T b + w^T c \geq 0$

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3. If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so

$$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$

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3. If $\lambda > 0$, then $\lambda c^T \geq (v^T - u^T)A \Rightarrow \lambda c^T w \geq (v^T - u^T)Aw$ and so

$$\lambda(u^T - v^T)b + \lambda w^T c \geq \lambda(u^T - v^T)b - (u^T - v^T)Aw$$

4. If $\lambda = 0$, let x, y be feasible solutions (which we assumed to exist). Then $x \geq 0, Ax = b$ and $y^TA \leq c^T$. Thus

$$c^Tw \geq y^TAw = 0 \geq (v^T - u^T)Ax = (v^T - u^T)b$$

Proof Strong Duality: $\lambda > 0$

Proof of Strong Duality: $\lambda = 0$

Affine form of Farkas' Lemma

A consequence of LP duality is the following lemma:

Lemma (Affine Farkas' Lemma)

Let the system

$$Ax \leq b$$

have at least one solution, and suppose that inequality

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holds whenever x *satisfies* $Ax \leq b$. *Then, for some* $\delta' \leq \delta$ *the linear inequality*

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Practice problem: use LP duality and Farkas' lemma to prove this lemma!

Complementary Slackness

- If the optima in both primal and dual is finite, and x, y are feasible solutions, the following are equivalent:
    1. x, y are optimal solutions to the primal and dual
    2. $c^T x = y^T b$
    3. if $x_i > 0$ then the corresponding inequality $y^T A_i \leq c_i$ is an equality: that is, we must have $y^T A_i = c_i$.

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- 2 and 3 are equivalent as we can write

$$c^T x - y^T b = c^T x - y^T A x = (c^T - y^T A)x = \sum_{i=1}^{n} (c_i - y^T A_i) x_i$$

Conclusion

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Today: *Linear Programming*

- Linear Programming and Duality - fundamental concepts, lots of applications!
    - Applications in Combinatorial Optimization (a lot of it happened here at UW!)
    - Applications in Game Theory (minimax theorem)
    - Applications in Learning Theory (boosting)
    - many more

Acknowledgement

- Lecture based largely on:
    - [Schrijver 1986, Chapter 7]

Proof of Fundamental Theorem of Linear Inequalities

Theorem (Farkas (1894, 1898), Minkowski (1896))

Let $a_1, \ldots, a_m, b \in \mathbb{R}^n$*, and* $t := \text{rank}\{a_1, \ldots, a_m, b\}$*. Then either*

1. *b is a non-negative linear combination of linearly independent vectors from $a_1, \ldots, a_m$, or*
2. *there exists a hyperplane $H := \{x \mid c^T x = 0\}$ s.t.*
    - $c^T b < 0$
    - $c^T a_i \geq 0$
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- Since 1 and 2 mutually exclusive, choose linearly independent $\mathcal{L}_0 := \{a_{i_1}, \ldots, a_{i_n}\}$
- We will perform an iterative procedure:

Proof of Fundamental Theorem of Linear Inequalities

Iterative procedure, starting with $\mathcal{L}_0$:

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- To conclude the proof, need to show that this procedure always terminates. If process doesn't terminate, there are two times $r < t$ such that $\mathcal{L}_r = \mathcal{L}_t$
- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.

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- Let ℓ be the highest index for which a_ℓ has been removed from $\mathcal{L}_k$ for some $r \leq k < t$.
- $\mathcal{L}_r = \mathcal{L}_t \Rightarrow a_\ell$ has also been added from some $\mathcal{L}_{k'}$ for some $r \leq k' < t$.

Proof of Fundamental Theorem of Linear Inequalities

- Say a_r was removed at iteration k and added back at iteration k' so $r \leq k < k' < t$
- Let c be the vector defining the hyperplane at the k' iteration (when we added a_r back to the set), and let $\mathcal{L}_k = \{a_{i_1}, \ldots, a_{i_n}\}$
- Now, above implies the following contradiction:

$$0 > c^T b = c^T(\lambda_{i_1} a_{i_1} + \cdots + \lambda_{i_n} a_{i_n}) = \lambda_{i_1} c^T a_{i_1} + \cdots \lambda_{i_n} c^T a_{i_n} \geq 0$$

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- Second inequality holds term by term:
  -

References I

Schrijver, Alexander (1986)
Theory of Linear and Integer Programming

Fourier, J. B. 1826
Analyse des travaux de l'Académie Royale des Sciences pendant l'année 1823.
Partie mathématique (1826)

Fourier, J. B. 1827
Analyse des travaux de l'Académie Royale des Sciences pendant l'année 1824.
Partie mathématique (1827)