Lecture 10: Fundamental Theorem of Markov Chains, Page Rank

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Overview

- Main Tools
  - Linear Algebra Background
  - Perron-Frobenius
- Main Applications
  - Fundamental Theorem of Markov Chains
  - Page Rank
- Conclusion
- Acknowledgements

Eigenvalues, Eigenvectors and Spectral Radius

- Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say that $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if there is a vector $v \in \mathbb{C}^n$ such that $Av = \lambda v$.

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- Example:

Positivity Lemma

Lemma (Positivity Lemma)

If $A \in \mathbb{R}^{n \times n}$ *is a positive matrix and* $u, v \in \mathbb{R}^n$ *are distinct vectors such that* $u \geq v$, *then* $Au > Av$. *Moreover, there exists* $\varepsilon > 0$ *such that* $Au > (1 + \varepsilon)Av$.

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- Note that

$$(A(u-v))_i = \sum_j A_{ij}(u_j - v_j) \geq (\min_{i,j} A_{ij}) \cdot \sum_j (u_j - v_j)$$

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- Since $u_j \geq v_j$ for all j and u, v distinct implies that there is one index k such that $u_k > v_k$, we have

$$\sum_j (u_j - v_j) \geq u_k - v_k > 0$$

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- the moreover part just follows from taking small enough ε.

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Let $A \in \mathbb{R}^{n \times n}$ be a positive matrix (i.e., all its entries are positive). Then, the following hold:

1. $\rho(A)$ *is an eigenvalue, and it has a positive eigenvector*
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- Let u be the vector defined by $u_i = |v_i|$. Then, we have

$$(Au)_i = \sum_j A_{ij} u_j \geq |\sum_j A_{ij} v_j| = |\lambda v_i| = \rho(A) \cdot u_i$$

so $Au \geq \rho(A)u$.

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- We proved $Au \geq \rho(A)u$.
- If inequality strict, then we have

$$A^2u > \rho(A) \cdot Au$$

and there is some positive $\varepsilon > 0$ such that

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$$A^n u \geq (1+\varepsilon)^n \cdot \rho(A)^n \cdot Au$$

- By Gelfand's formula we would have

$$\rho(A) = \lim_{n\to\infty} \|A^n\|_F^{1/n} \geq (1+\varepsilon)\rho(A)$$

which is a contradiction. So equality must hold.

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- If we had another eigenvalue $\lambda \neq \rho(A)$ in the circumference $|\mu| = \rho(A)$, where z is the eigenvector corresponding to λ, by the previous slide, we know that w defined as $w_i = |z_i|$ satisfies

$$Aw = \rho(A)w \quad \Leftrightarrow \quad \sum_j A_{ij}w_j = \rho(A) \cdot |z_i| = |\lambda z_i| = |\sum_j A_{ij}z_j|$$

for every $1 \leq i \leq n$

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  for every $1 \leq i \leq n$
- *Lemma*: if the conditions above hold, then there is $\alpha \in \mathbb{C}$ nonzero such that $\alpha z \geq 0$

  Proof by squaring both sides and using complex conjugates.

Perron's theorem - items 2 and 3

- But if $\alpha z \geq 0$ and a nonzero vector, we have

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- Let $\beta > 0$ be such that $u - \beta v \geq 0$ and at least one entry is zero.
- $u - \beta v \neq 0$ since the vectors are linearly independent
- But for each $1 \leq i \leq n$

$$\rho(A) \cdot (u - \beta v)_i = (A(u - \beta v))_i > 0$$

  which contradicts our choice of β. Thus, there cannot be two linearly independent eigenvectors.

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Theorem (Perron-Frbenius)

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- By previous lecture, we saw that A being aperiodic and irreducible implies that there is $m > 0$ such that A^m has all positive entries.
- Apply Perron's theorem to A^m and note that the eigenvalues of A^m are λ_i^m, where λ_i are the eigenvalues of A

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Theorem (Fundamental Theorem of Markov Chains)

Any finite, irreducible and aperiodic Markov Chain has the following properties:

1. *There exists a unique stationary distribution* π, *where* $\pi_i > 0$ *for all* $i \in [n]$
2. *The sequence of distributions* $\{p_t\}_{t \geq 0}$ *will converge to* π, *no matter what the initial distribution is*
3. $$\pi_i = \lim_{t \to \infty} P^t_{i,i} = \frac{1}{\tau_{i,i}}$$

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- The transition matrix P is non-negative, irreducible and aperiodic. So we can apply Perron-Frobenius and prove items 1 and 2.

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If our underlying graph is undirected:

- If A_G adjacency matrix of $G(V,E)$ and $D = diag(d_1, d_2, \ldots, d_n)$, transition matrix:

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    - *unique* eigenvector whose eigenvalue has *max absolute value*
    - eigenvector has *all positive coordinates*
    - eigenvalue is *positive*

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    - eigenvalue is *positive*
  - This eigenvector is π!

Fundamental Theorem of Markov Chains

- Stationary distribution: $\pi_i = \dfrac{d_i}{2m}, \quad m = |E|$
- Transition matrix: $P = D^{-1} \cdot A_G$
  - P *similar* to a symmetric matrix: $P' = D^{-1/2} A_G D^{-1/2}$
  - P and P' has *same eigenvalues*! And P' has only *real eigenvalues*!
  - Eigenvectors of P are $D^{-1/2} v_i$ where v_i are eigenvectors of P'. And v_i can be taken to form *orthonormal basis*.
  - Graph strongly connected $\Rightarrow$ *Perron-Frobenius* for irreducible non-negative matrices
    - *unique* eigenvector whose eigenvalue has *max absolute value*
    - eigenvector has *all positive coordinates*
    - eigenvalue is *positive*
  - This eigenvector is π!
  - All random walks converge to π, as we wanted to show.

- Main Tools
  - Linear Algebra Background
  - Perron-Frobenius
- Main Applications
  - Fundamental Theorem of Markov Chains
  - Page Rank
- Conclusion
- Acknowledgements

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- If graph finite, irreducible and aperiodic, fundamental theorem guarantees stationary distribnution.

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- This modification does not change "relative importance" of vertices

Conclusion

Markov Chains and Random Walks are ubiquitous in randomized algorithms.

- Page Rank algorithm (today's lecture)
- Approximation algorithms for counting problems [Karp, Luby & Madras]
    - Permanent of non-negative matrices [Jerrum, Vigoda & Sinclair]
- Sampling Problems
    - Gibbs sampling in statistical physics
    - many more places
- Probability amplification without too much randomness (efficient)
    - Random walks on expander graphs
- many more

Acknowledgement

- Lecture based largely on:
  - Hannah Cairns notes on Perron-Frobenius (see link in course webpage)
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 6]
- See Lap Chi's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L11.pdf
- Also see Lap Chi's notes https://cs.uwaterloo.ca/~lapchi/cs466/notes/L14.pdf for a proof of fundamental theorem of Markov chains for undirected graphs.

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