Lecture 07: Algebraic Techniques
Fingerprinting, Verifying Polynomial Identities, Parallel Algorithms for Matching Problems

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We The Champions!

https://scoreboard.nac.icpc.global/

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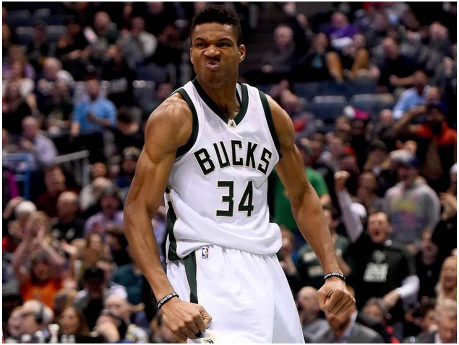

Figure: That's right

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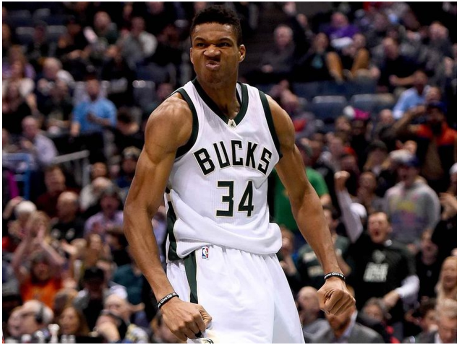

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So Homework will be more fun from now on :)

Overview

- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
- Acknowledgements

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Communication complexity setting, randomized algorithms, need to work with high probability.

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- what happens when they are different?

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- Number of bits sent is $\tilde{O}(\log t + \log n)$. Choosing $t = n$ solves it.

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Lemma (Roots of Univariate Polynomials)

Let $\mathbb{F}$ be a field and $P(x) \in \mathbb{F}[x]$ be a nonzero univariate polynomial of degree d. Then $P(x)$ has at most d distinct roots in $\overline{\mathbb{F}}$.

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- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

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- Let $Q(x) = P_3(x) - P_1(x) \cdot P_2(x)$. It has degree $\leq 2n$
- By lemma, if $Q \neq 0$ then $Q(a) = 0$ for at most $2n$ values in $\mathbb{F}$.
- Take a set $S \subseteq \mathbb{F}$ of size $4n$. Let $a \in S$ chosen randomly.
- Compute $Q(a)$ by computing $P_1(a), P_2(a), P_3(a)$ and then $P_3(a) - P_1(a) \cdot P_2(a)$
- Probability $Q(a) = 0$ (i.e., we failed to identify non-zero)

$$\leq \frac{\deg(Q)}{|S|} \leq \frac{2n}{4n} = 1/2.$$

- Can amplify probability by running multiple times or by choosing larger set S.

Polynomial Identity Testing

Lemma (Ore-Schwartz-Zippel-de Millo-Lipton lemma)

Let $\mathbb{F}$ *be a field and* $P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ *be a* *nonzero* *polynomial of degree* $\leq d$*. Then for any set* $S \subseteq \overline{\mathbb{F}}$*, we have:*

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Proof by induction in number of variables.

Bipartite Matching

- **Input:** bipartite graph $G(L, R, E)$ with $|L| = |R| = n$
- **Output:** does G have a perfect matching?

[2] First proved by Edmonds.

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- **Algorithm:** evaluate $\det(X)$ at a random value for the variables $y_{i,j}$.

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$$[T_G]_{i,j} = \begin{cases} x_{i,j} & \text{if } (i,j) \in F \\ -x_{i,j} & \text{if } (j,i) \in F \\ 0 & \text{otherwise} \end{cases}$$

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- Each permutation $\sigma \in S_{2n}$ that yields non-zero term corresponds to a (directed) subgraph of G $H_\sigma(V, F_\sigma)$, where $F_\sigma = \{(i, \sigma(i))\}_{i=1}^{2n}$.

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- Each vertex in H_σ has $|\delta^{out}(i)| = |\delta^{in}(i)| = 1$.

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- If σ only has even cycles, then H_σ gives us a perfect matching (by taking every other edge of the graph H_σ, ignoring orientation)

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- Otherwise, for each $\sigma \in S_{2n}$ (that has odd cycle), there is another permutation $r(\sigma) \in S_{2n}$ that is obtained by reversing odd cycle of H_σ containing vertex with *minimum index*.

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- Is there a term that does not cancel? (have to show that $\det(T_G) \not\equiv 0$)
- If T_G has a matching, say, $\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}$, then take permutation $\sigma = (1\ 2)(3\ 4)\cdots(2n-1\ 2n)$

$$(-1)^{\sigma}\prod_{i=1}^{2n}[T_G]_{i,\sigma(i)} = (-1)^n\prod_{i=1}^{n} -x^2_{(2i-1)\sigma(2i-1)} = \prod_{i=1}^{n} x^2_{(2i-1)\sigma(2i-1)}.$$

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We have seen randomized algorithms for bipartite and non-bipartite matching.
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- Introduction
  - Why Algebraic Techniques in computer science?
  - Fingerprinting: String equality verification
- Main Problems
  - Polynomial Identity Testing
  - Randomized Matching Algorithms
  - Isolation Lemma
- Remarks
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Remark

The isolation lemma could be quite counter-intuitive. A set system can have $\Omega(2^n)$ sets. On average, there are $\Omega(2^n/(2n^2))$ sets of a given weight, as max weight is $\leq 2n^2$. Isolation lemma tells us that with high probability there is *only one* set of minimum weight.

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9. If two different sets A, B have minimum weight, then any element in $A \Delta B$ must be ambiguous.
10. Probability that this happens is $\leq 1/2$. (step 8)

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Remarks

It is hard to overstate the importance of algebraic techniques in computing.

- Very useful tool for randomized algorithms (hashing, today's lecture)
- Parallel & Distributed Computing (this lecture and lectures 21 and 23)
- Interactive proof systems
- Efficient proof/program verification (PCP - a bit in lecture 16)
    - Applications in hardness of approximation!
    - Applications in blockchain (Zcash for instance)
    - Zero Knowledge proofs (lecture 24)
- Cryptography
- Coding theory
- many more...

Derandomizing (i.e., obtaining deterministic algorithms) for some of these settings (whenever possible) is *major open problem* in computer science.

Potential Final Projects

- Can we derandomize the perfect matching algorithms from class?
- A lot of progress has been made in the past couple years on this question in the works [Fenner, Gurjar & Thierauf 2019] and subsequently [Svensson & Tarnawski 2017]
- Survey of the above, or understanding these papers is a great final project!

Acknowledgement

- Lecture based largely on:
  - Lap Chi's notes
  - [Motwani & Raghavan 2007, Chapter 7]
  - [Korte & Vygen 2012, Chapter 10].
- See Lap Chi's notes at
  `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L07.pdf`

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