Lecture 4: Balls & Bins

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

May 18, 2023

Overview

- Introduction
  - Probability basic notions
  - Balls and Bins
  - Analyses
- Coupon Collector and Power of Two Choices
  - Coupon Collector
  - Power of Two Choices
- Acknowledgements

Event Spaces and Inclusion-Exclusion

Union Bound and Inclusion-Exclusion

Union Bound and Inclusion-Exclusion

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is

$$\Pr[E_1 \mid E_2] := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is

$$\Pr[E_1 \mid E_2] := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$

- If $E_1, \ldots, E_k$ partition our sample space, then for any event E

$$\Pr[E] = \sum_{i=1}^{k} \Pr[E \mid E_i] \cdot \Pr[E_i]$$

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is

$$\Pr[E_1 \mid E_2] := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$

- If $E_1, \ldots, E_k$ partition our sample space, then for any event E

$$\Pr[E] = \sum_{i=1}^{k} \Pr[E \mid E_i] \cdot \Pr[E_i]$$

- **Simple Bayes' rule**:

$$\Pr[E_1 \mid E_2] = \frac{\Pr[E_2 \mid E_1] \cdot \Pr[E_1]}{\Pr[E_2]}$$

Conditional Probability and Bayes Rule

- The *conditional probability* of E_1 given E_2 is

$$\Pr[E_1 \mid E_2] := \frac{\Pr[E_1 \cap E_2]}{\Pr[E_2]}$$

- If $E_1, \ldots, E_k$ partition our sample space, then for any event E

$$\Pr[E] = \sum_{i=1}^{k} \Pr[E \mid E_i] \cdot \Pr[E_i]$$

- **Simple Bayes' rule**:

$$\Pr[E_1 \mid E_2] = \frac{\Pr[E_2 \mid E_1] \cdot \Pr[E_1]}{\Pr[E_2]}$$

- **Bayes' rule**: $E_1, \ldots, E_k$ partition our sample space then for event E

$$\Pr[E_i \mid E] = \frac{\Pr[E \cap E_i]}{\Pr[E]} = \frac{\Pr[E \mid E_i] \cdot \Pr[E_i]}{\sum_{j=1}^{k} \Pr[E \mid E_j] \cdot \Pr[E_j]}$$

Balls and Bins Questions

Setup: we have m balls and we want to put them in n bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

Balls and Bins Questions

Setup: we have *m* balls and we want to put them in *n* bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

Balls and Bins Questions

Setup: we have *m* balls and we want to put them in *n* bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

- What is the *expected* number of balls in a bin?

Balls and Bins Questions

Setup: we have m balls and we want to put them in n bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

- What is the *expected* number of balls in a bin?
- What is the *expected* number of empty bins?

Balls and Bins Questions

Setup: we have *m* balls and we want to put them in *n* bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

- What is the *expected* number of balls in a bin?
- What is the *expected* number of empty bins?
- What is "typically" the *maximum* number of balls in any bin?

Balls and Bins Questions

Setup: we have *m* balls and we want to put them in *n* bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

- What is the *expected* number of balls in a bin?
- What is the *expected* number of empty bins?
- What is "typically" the *maximum* number of balls in any bin?
- What is the *expected* number of bins with k balls in them?

Balls and Bins Questions

Setup: we have m balls and we want to put them in n bins.

We will do this by throwing each ball into a *uniformly random* bin *independently*.

We are interested in the following questions:

- What is the *expected* number of balls in a bin?
- What is the *expected* number of empty bins?
- What is "typically" the *maximum* number of balls in any bin?
- What is the *expected* number of bins with k balls in them?
- For what values of m do we expect to have *no empty bins*? (coupon collector)

Why Learn About Balls and Bins?

In **this lecture**, we will analyse random processes (*balls & bins*) which underlie several randomized algorithms!

Applications ranging from:

1. data structures
2. routing in parallel computers
3. many more!

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$\mathbb{E}[\#\text{ balls in bin } j] = \mathbb{E}\left[\sum_{i=1}^{m} B_{ij}\right]$$

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$\begin{aligned}
\mathbb{E}[\#\text{ balls in bin } j] &= \mathbb{E}\left[\sum_{i=1}^{m} B_{ij}\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[B_{ij}\right] && \text{(linearity of expectation)}
\end{aligned}$$

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$\begin{aligned}
\mathbb{E}[\#\text{ balls in bin } j] &= \mathbb{E}\left[\sum_{i=1}^{m} B_{ij}\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[B_{ij}\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{m} \Pr\left[\text{ball } i \text{ in bin } j\right]
\end{aligned}$$

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$
\begin{aligned}
\mathbb{E}[\#\text{ balls in bin } j] &= \mathbb{E}\left[\sum_{i=1}^{m} B_{ij}\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[B_{ij}\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{m} \Pr\left[\text{ball } i \text{ in bin } j\right] \\
&= \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n} && \text{(uniformly at random)}
\end{aligned}
$$

Expected Number of Balls in a Bin

Let us label the m balls $1, \ldots, m$, and the n bins $1, 2, \ldots, n$.
Let B_{ij} be the indicator variable that ball i was thrown into bin j.

$$\begin{aligned}
\mathbb{E}[\#\text{ balls in bin } j] &= \mathbb{E}\left[\sum_{i=1}^{m} B_{ij}\right] \\
&= \sum_{i=1}^{m} \mathbb{E}\left[B_{ij}\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{m} \Pr\left[\text{ball } i \text{ in bin } j\right] \\
&= \sum_{i=1}^{m} \frac{1}{n} = \frac{m}{n} && \text{(uniformly at random)}
\end{aligned}$$

When $m = n$, expectation of one ball per bin. How often will this actually happen?

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$\mathbb{E}[\#\text{ empty bins}] = \mathbb{E}\left[\sum_{i=1}^{n} N_i\right]$$

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$\begin{aligned}\mathbb{E}[\#\text{ empty bins}] &= \mathbb{E}\left[\sum_{i=1}^{n} N_i\right] \\ &= \sum_{i=1}^{n} \mathbb{E}\left[N_i\right] && \text{(linearity of expectation)}\end{aligned}$$

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$\begin{aligned}
\mathbb{E}[\#\text{ empty bins}] &= \mathbb{E}\left[\sum_{i=1}^{n} N_i\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[N_i\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{n} \Pr\left[\text{bin } i \text{ is empty}\right]
\end{aligned}$$

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$
\begin{aligned}
\mathbb{E}[\#\text{ empty bins}] &= \mathbb{E}\left[\sum_{i=1}^{n} N_i\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[N_i\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{n} \Pr\left[\text{bin } i \text{ is empty}\right] \\
&= \sum_{i=1}^{n} (1 - 1/n)^m \\
&= n \cdot (1 - 1/n)^m \approx n \cdot e^{-m/n}
\end{aligned}
$$

Expected number of empty bins

Let N_i be the indicator variable that bin i is empty.

$$\begin{aligned}
\mathbb{E}[\#\text{ empty bins}] &= \mathbb{E}\left[\sum_{i=1}^{n} N_i\right] \\
&= \sum_{i=1}^{n} \mathbb{E}\left[N_i\right] && \text{(linearity of expectation)} \\
&= \sum_{i=1}^{n} \Pr\left[\text{bin } i \text{ is empty}\right] \\
&= \sum_{i=1}^{n} (1 - 1/n)^m \\
&= n \cdot (1 - 1/n)^m \approx n \cdot e^{-m/n}
\end{aligned}$$

When $m = n$, expected fraction of empty bins is $\frac{1}{e}$.

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

When $m = n$, second calculation had expectation of $1/e$ *fraction of empty bins*.

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

When $m = n$, second calculation had expectation of $1/e$ *fraction of empty bins*.

Which expectation should I actually "expect"?

Head Scratching Moment

When $m = n$, first calculation had expectation of *one ball per bin*.

When $m = n$, second calculation had expectation of $1/e$ *fraction of empty bins*.

Which expectation should I actually "expect"?

As we mentioned earlier, this is where *concentration of probability measure* tries to address. It turns out that the *second random variable* (and thus second calculation) is concentrated around the mean (i.e., expectation).

So we "expect" (or it is "typical") to see around $1/e$-fraction of empty bins when $m = n$

Maximum load in a bin

What is the "typical" maximum number of balls in a bin?

As we saw in the previous slide, "typical" is related to concentration of probability measure.

Maximum load in a bin

What is the "typical" maximum number of balls in a bin?

As we saw in the previous slide, "typical" is related to concentration of probability measure.

Let us first see a simpler problem, which is known as the *birthday paradox*: for what value of m do we expect to see two balls in one bin?

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \cdot \left(1 - \frac{m-1}{n}\right) \leq e^{-1/n} \cdots e^{-\frac{m-1}{n}} \approx e^{\frac{-m^2}{2n}}$$

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \cdot \left(1 - \frac{m-1}{n}\right) \leq e^{-1/n} \cdots e^{-\frac{m-1}{n}} \approx e^{\frac{-m^2}{2n}}$$

This is $\leq 1/2$ when $m = \sqrt{2n \ln(2)}$. For $n = 365$, this is $m \approx 22.4$ for the probability that two people *(balls)* have birthday on the same date *(bins)* to become $\geq 1/2$.

Birthday Paradox

The probability that there are no collisions after we have thrown m balls is:

$$1 \cdot \left(1 - \frac{1}{n}\right) \cdot \left(1 - \frac{2}{n}\right) \cdots \cdot \left(1 - \frac{m-1}{n}\right) \leq e^{-1/n} \cdots e^{-\frac{m-1}{n}} \approx e^{\frac{-m^2}{2n}}$$

This is $\leq 1/2$ when $m = \sqrt{2n \ln(2)}$. For $n = 365$, this is $m \approx 22.4$ for the probability that two people *(balls)* have birthday on the same date *(bins)* to become $\geq 1/2$.

Thus, expect to see collision (two balls in the same bin) when $m = \Theta(\sqrt{n})$. This appears in several places:

- hashing
- factoring
- many more

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

$$\Pr[\text{bin 1 has } \geq k \text{ balls}] \leq \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}]$$

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

$$\Pr[\text{bin 1 has } \geq k \text{ balls}] \leq \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}]$$

$$= \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \frac{1}{n}$$

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

$$\begin{aligned}
\Pr[\text{bin 1 has } \geq k \text{ balls}] &\leq \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}] \\
&= \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \frac{1}{n} \\
&= \binom{n}{k} \cdot \frac{1}{n^k} \leq \left(\frac{ne}{k}\right)^k \cdot \frac{1}{n^k} = \frac{e^k}{k^k}
\end{aligned}$$

Maximum load in a bin when $m = n$

What is the probability that a particular bin (say bin 1) has $\geq k$ balls in it?

$$\Pr[\text{bin 1 has } \geq k \text{ balls}] \leq \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \Pr[\text{ball } i \text{ in bin 1}]$$

$$= \sum_{\substack{S \ subset[n] \\ |S|=k}} \prod_{i \in S} \frac{1}{n}$$

$$= \binom{n}{k} \cdot \frac{1}{n^k} \leq \left(\frac{ne}{k}\right)^k \cdot \frac{1}{n^k} = \frac{e^k}{k^k}$$

By union bound

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq \sum_{i=1}^{n} \Pr[\text{bin i has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k}$$

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

When will the above probability be large (say $>> 1/2$)?

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

When will the above probability be large (say $>> 1/2$)?

When $k \ln k > \ln n$. Setting $k = 3\frac{\ln n}{\ln \ln n}$ does it.

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

When will the above probability be large (say $>> 1/2$)?

When $k \ln k > \ln n$. Setting $k = 3\frac{\ln n}{\ln \ln n}$ does it.

With high probability, max load is $O\left(\frac{\ln n}{\ln \ln n}\right)$.

Maximum load in a bin when $m = n$

$$\Pr[\text{some bin has } \geq k \text{ balls}] \leq n \cdot \frac{e^k}{k^k} = e^{\ln n + k - k \ln k}$$

$$\Pr[\text{max load is } \leq k] = 1 - \Pr[\text{some bin has } > k \text{ balls}] \geq 1 - e^{\ln n + k - k \ln k}$$

When will the above probability be large (say $>> 1/2$)?

When $k \ln k > \ln n$. Setting $k = 3 \frac{\ln n}{\ln \ln n}$ does it.

With high probability, max load is $O\left(\frac{\ln n}{\ln \ln n}\right)$.

This comes up in hashing and in analysis of approximation algorithms (for instance, best known approximation ratio for congestion minimization).

Coupon Collector

For what value of m do we expect to have no empty bins?

Coupon Collector

For what value of m do we expect to have no empty bins?

Why is this problem called the coupon collector problem?

Because we can formulate it in the following way:

- suppose each bin is a different coupon
- we buy one coupon at random (like kinder eggs/pack action cards)
- what is the number of coupons that we need to buy to collect all of them?

Coupon Collector

For what value of m do we expect to have no empty bins?

Why is this problem called the coupon collector problem?

Because we can formulate it in the following way:

- suppose each bin is a different coupon
- we buy one coupon at random (like kinder eggs/pack action cards)
- what is the number of coupons that we need to buy to collect all of them?

Let X_i be the number of balls thrown to get from i empty bins to $i-1$ empty bins. Let X be the number of balls thrown until we have no empty bins.

$$X = \sum_{i=1}^{n} X_i$$

Coupon Collector

- $X_i \leftarrow \#$ balls thrown to get from i empty bins to $i-1$ empty bins
- $X \leftarrow \#$ balls thrown until we have no empty bins

Coupon Collector

- $X_i \leftarrow \#$ balls thrown to get from i empty bins to $i-1$ empty bins
- $X \leftarrow \#$ balls thrown until we have no empty bins

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

Coupon Collector

- $X_i \leftarrow \#$ balls thrown to get from i empty bins to $i-1$ empty bins
- $X \leftarrow \#$ balls thrown until we have no empty bins

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i]$$

What is $\mathbb{E}[X_i]$?

Coupon Collector

- $X_i \leftarrow$ # balls thrown to get from i empty bins to $i-1$ empty bins
- $X \leftarrow$ # balls thrown until we have no empty bins

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[X_i\right]$$

What is $\mathbb{E}[X_i]$?

X_i geometric random variable with parameter $p = \frac{i}{n}$.

Number of trials until the first success, where success probability p.

$$\Pr[X_i = k] = (1-p)^{k-1} \cdot p$$

Coupon Collector - Computing $\mathbb{E}[X]$

Coupon Collector - Computing $\mathbb{E}[X]$

This $n \ln n$ bound shows up in:

- cover time of random walks in complete graph
- number of edges needed in graph sparsification
- many more places

Power of Two Choices

We now know that when n balls are thrown into n bins, the maximum load is $\Theta(\ln n / \ln \ln n)$ with constant probability.[1]

[1] we'll maybe see lower bound later

Power of Two Choices

We now know that when n balls are thrown into n bins, the maximum load is $\Theta(\ln n / \ln \ln n)$ with constant probability.[1]

Consider following variant: what if when throwing a ball in a bin, *before* we throw the ball we choose *two* bins *uniformly at random* and put the ball in the *bin with fewer balls*?

[1] we'll maybe see lower bound later

Power of Two Choices

We now know that when n balls are thrown into n bins, the maximum load is $\Theta(\ln n / \ln \ln n)$ with constant probability.[1]

Consider following variant: what if when throwing a ball in a bin, *before* we throw the ball we choose *two* bins *uniformly at random* and put the ball in the *bin with fewer balls*?

This simple modification reduces maximum load to $O(\ln \ln n)$!

[1] we'll maybe see lower bound later

Power of Two Choices

We now know that when n balls are thrown into n bins, the maximum load is $\Theta(\ln n / \ln \ln n)$ with constant probability.[1]

Consider following variant: what if when throwing a ball in a bin, *before* we throw the ball we choose *two* bins *uniformly at random* and put the ball in the *bin with fewer balls*?

This simple modification reduces maximum load to $O(\ln \ln n)$!

Intuition/idea: let the height of a bin be the # balls in that bin. This process tells us that to get one bin with height $h + 1$ we must have at least two bins of height h.

We can bound # bins with height at least h (because this will tell us how likely it is to get to height $h + 1$).

[1] we'll maybe see lower bound later

A bit more intuition

$N_h :=$ number of bins with height at least h

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)
- Probability of selecting two such bins is $1/16$

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)
- Probability of selecting two such bins is $1/16$
- So we should expect only $n/16$ bins with height 5
- And only $n/256 = n/16^2 = n/2^{2^3}$ bins with height 6

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \le \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)
- Probability of selecting two such bins is $1/16$
- So we should expect only $n/16$ bins with height 5
- And only $n/256 = n/16^2 = n/2^{2^3}$ bins with height 6
- Repeating this, we should expect $\dfrac{n}{2^{2^{h-3}}}$ bins of height h

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)
- Probability of selecting two such bins is $1/16$
- So we should expect only $n/16$ bins with height 5
- And only $n/256 = n/16^2 = n/2^{2^3}$ bins with height 6
- Repeating this, we should expect $\frac{n}{2^{2^{h-3}}}$ bins of height h
- So expect $\log \log n$ maximum height after throwing n balls.

A bit more intuition

$N_h :=$ number of bins with height at least h

$$\Pr[\text{at least one bin of height } h+1] \leq \left(\frac{N_h}{n}\right)^2$$

- Say we have only $n/4$ bins with 4 items (i.e. height 4)
- Probability of selecting two such bins is $1/16$
- So we should expect only $n/16$ bins with height 5
- And only $n/256 = n/16^2 = n/2^{2^3}$ bins with height 6
- Repeating this, we should expect $\frac{n}{2^{2^{h-3}}}$ bins of height h
- So expect $\log \log n$ maximum height after throwing n balls.

How do we turn this into a proof?
See [Mitzenmacher & Upfal, Chapter 14] and Lap Chi's notes.

Acknowledgement

- Lecture based largely on Lap Chi's notes and on [Motwani & Raghavan 2007, Chapter 3].
- See Lap Chi's notes at https://cs.uwaterloo.ca/~lapchi/cs466/notes/L04.pdf

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Mitzenmacher, Michael, and Eli Upfal (2017)
Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis.
Cambridge university press, 2017.