Lecture 3: Concentration Inequalities

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

May 18, 2023

Overview

- Introduction
  - Concentration Inequalities
  - Markov's Inequality
- Higher Moments
  - Moments and Variance
  - Chebyshev's Inequality
  - Chernoff-Hoeffding's Inequality
- Acknowledgements

Why do we want concentration?

When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

"our algorithm will run in time $\approx T$ *very often*."

Why do we want concentration?

When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

"our algorithm will run in time $\approx T$ *very often*."

That is,

- not only analyse the *expected running times* of the algorithms,
- we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

Why do we want concentration?

When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

"our algorithm will run in time $\approx T$ *very often*."

That is,

- not only analyse the *expected running times* of the algorithms,
- we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

Running time *small* with *high probability* *better than* small expected running time.

Why do we want concentration?

When evaluating performance of randomized algorithms, not enough to know our algorithm runs in expected time T. What we want to say is

"our algorithm will run in time $\approx T$ *very often*."

That is,

- not only analyse the *expected running times* of the algorithms,
- we would also like to know if the algorithm runs in time close to its expected running time *most of the time*.

Running time *small* with *high probability* *better than* small expected running time.

Often times in algorithm analysis, running time is *concentrated* around expectation. This *concentration of measure* proves that our algorithms will *typically* run in time close to expectation.

Today's inequalities

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

Today's inequalities

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

Theorem (Chebyshev's Inequality)

Let X be a discrete random variable. Then

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}, \quad \forall t > 0.$$

Today's inequalities II

Theorem (Chernoff-Hoeffding's Inequality)

Let $X_1, \ldots, X_n$ *be independent indicator variables such that* $\Pr[X_i = 1] = p_i$*, where* $0 < p_i < 1$*. Let* $X = \sum_{i=1}^{n} X_i$ *and* $\delta > 0$*. Then*

$$\Pr[X \geq (1+\delta) \cdot \mathbb{E}[X]] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mathbb{E}[X]},$$

and

$$\Pr[X \leq (1-\delta) \cdot \mathbb{E}[X]] \leq \exp\left(-\mathbb{E}[X] \cdot \delta^2/2\right).$$

Markov's Inequality

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

Markov's Inequality

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

- **Quicksort:** Expected running time of Quicksort is $2n \ln n$. Markov's inequality tells us that the runtime is at least $2cn \ln n$ with probability $\leq 1/c$, for any $c \geq 1$

Markov's Inequality

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

- **Quicksort:** Expected running time of Quicksort is $2n \ln n$. Markov's inequality tells us that the runtime is at least $2cn \ln n$ with probability $\leq 1/c$, for any $c \geq 1$
- **Coin Flipping:** If we flip n fair coins, the expected number of heads is $n/2$. Markov's inequality tells us that $\Pr[\text{ \# heads } \geq 3n/4] \leq 2/3$

Markov's Inequality

Theorem (Markov's Inequality)

Let X be a non-negative discrete random variable. Then

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}, \quad \forall t > 0.$$

- **Quicksort:** Expected running time of Quicksort is $2n \ln n$. Markov's inequality tells us that the runtime is at least $2cn \ln n$ with probability $\leq 1/c$, for any $c \geq 1$
- **Coin Flipping:** If we flip n fair coins, the expected number of heads is $n/2$. Markov's inequality tells us that $\Pr[\ \#\ \text{heads}\ \geq 3n/4] \leq 2/3$

Remark

Useful when we have no information beyond expected value (or when random variable difficult to analyze). Otherwise other inequalities much sharper!

Markov's Inequality

Some practice problems.

- Is Markov's inequality tight? Can you give an example?

Markov's Inequality

Some practice problems.

- Is Markov's inequality tight? Can you give an example?
- Does it hold for general random variables (not just non-negative)?

Markov's Inequality

Some practice problems.

- Is Markov's inequality tight? Can you give an example?
- Does it hold for general random variables (not just non-negative)?
- Can it be modified to upper bound $\Pr[X \leq t]$?

- Introduction
    - Concentration Inequalities
    - Markov's Inequality
- Higher Moments
    - Moments and Variance
    - Chebyshev's Inequality
    - Chernoff-Hoeffding's Inequality
- Acknowledgements

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).

How to distinguish between:

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).
How to distinguish between:

- X such that $\Pr[X = i] = \begin{cases} 1/n, & \text{if } 1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).

How to distinguish between:

- X such that $\Pr[X = i] = \begin{cases} 1/n, & \text{if } 1 \le i \le n \\ 0, & \text{otherwise} \end{cases}$
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$
- same expectation, but very different random variables...

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).

How to distinguish between:

- X such that $\Pr[X = i] = \begin{cases} 1/n, & \text{if } 1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$
- same expectation, but very different random variables...
- Look at how far variable usually is from its expectation. How to do that?

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).
How to distinguish between:

- X such that $\Pr[X = i] = \begin{cases} 1/n, & \text{if } 1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$
- same expectation, but very different random variables...
- Look at how far variable usually is from its expectation. How to do that?
- How to bound $\Pr[|X - \mathbb{E}[X]| \geq t]$?

Moments and Variance

To give better bounds, we need more information about the random variable (beyond expected value).
How to distinguish between:

- X such that $\Pr[X = i] = \begin{cases} 1/n, & \text{if } 1 \leq i \leq n \\ 0, & \text{otherwise} \end{cases}$
- Y such that $\Pr[Y = 1] = 1/2$ and $\Pr[Y = n] = 1/2$
- same expectation, but very different random variables...
- Look at how far variable usually is from its expectation. How to do that?
- How to bound $\Pr[|X - \mathbb{E}[X]| \geq t]$?

Theorem (Chebyshev's Inequality)

Let X be a discrete random variable. Then

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}, \quad \forall t > 0.$$

Chebyshev's inequality

Let X be a random variable.

Chebyshev's inequality

Let X be a random variable.

- Its **Variance** is defined as $\mathrm{Var}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2]$
- and its **standard deviation** is $\sigma(X) := \sqrt{\mathrm{Var}[X]}$

Chebyshev's inequality

Let X be a random variable.

- Its **Variance** is defined as $\mathrm{Var}[X] := \mathbb{E}[(X - \mathbb{E}[X])^2]$
- and its **standard deviation** is $\sigma(X) := \sqrt{\mathrm{Var}[X]}$

Theorem (Chebyshev's Inequality)

Let X be a discrete random variable. Then

$$\Pr[|X - \mathbb{E}[X]| \geq t] \leq \frac{\mathrm{Var}[X]}{t^2}, \quad \forall t > 0.$$

Covariance

How do we measure the correlation between two random variables?

Covariance

How do we measure the correlation between two random variables?

Definition (Covariance)

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X, Y] := \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])].$$

We say that X, Y are *positively correlated* if $\mathrm{Cov}[X, Y] > 0$ and *negatively correlated* if $\mathrm{Cov}[X, Y] < 0$.

Covariance

How do we measure the correlation between two random variables?

Definition (Covariance)

The *covariance* of two random variables X, Y is defined as

$$\mathrm{Cov}[X,Y] := \mathbb{E}[(X - \mathbb{E}[X]) \cdot (Y - \mathbb{E}[Y])].$$

We say that X, Y are *positively correlated* if $\mathrm{Cov}[X,Y] > 0$ and *negatively correlated* if $\mathrm{Cov}[X,Y] < 0$.

Proposition

- $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X,Y]$
- *If* X, Y *are independent, then* $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$

Chebyshev & Covariance example

Coin Flipping: If X be $\#$ heads in n independent unbiased coin flips, let us bound again $\Pr[X \geq 3n/4]$.

Chebyshev & Covariance example

Coin Flipping: If X be $\#$ heads in n independent unbiased coin flips, let us bound again $\Pr[X \geq 3n/4]$.

- $X_i = \begin{cases} 1, & \text{if coin flipped heads} \\ 0, & \text{otherwise} \end{cases}$
- $X = \sum_{i=1}^{n} X_i$, and we know that X_i, X_j are independent

Chebyshev & Covariance example

Coin Flipping: If X be # heads in n independent unbiased coin flips, let us bound again $\Pr[X \geq 3n/4]$.

- $X_i = \begin{cases} 1, & \text{if coin flipped heads} \\ 0, & \text{otherwise} \end{cases}$
- $X = \sum_{i=1}^{n} X_i$, and we know that X_i, X_j are independent
- By proposition:

$$\text{Var}[X] = \sum_{i=1}^{n} \text{Var}[X_i] = n/4$$

Chebyshev & Covariance example

Coin Flipping: If X be # heads in n independent unbiased coin flips, let us bound again $\Pr[X \geq 3n/4]$.

- $X_i = \begin{cases} 1, & \text{if coin flipped heads} \\ 0, & \text{otherwise} \end{cases}$
- $X = \sum_{i=1}^{n} X_i$, and we know that X_i, X_j are independent
- By proposition:

$$\mathrm{Var}[X] = \sum_{i=1}^{n} \mathrm{Var}[X_i] = n/4$$

- Chebyshev:

$$\Pr[X \geq 3n/4] \leq \Pr[|X - n/2| \geq n/4] \leq \frac{n/4}{(n/4)^2} = 4/n$$

Higher Moments

To obtain even more information of a random variable, useful to see more of its moments:

Higher Moments

To obtain even more information of a random variable, useful to see more of its moments:

- the *k^{th} moment* of random variable X is $\mathbb{E}[X^k]$.

Higher Moments

To obtain even more information of a random variable, useful to see more of its moments:

- the *k^{th} moment* of random variable X is $\mathbb{E}[X^k]$.
- the *k^{th} central moment* of random variable X is

$$\mu_X^{(k)} := \mathbb{E}[(X - \mathbb{E}[X])^k],$$

  if it exists.

Higher Moments

To obtain even more information of a random variable, useful to see more of its moments:

- the k^{th} *moment* of random variable X is $\mathbb{E}[X^k]$.
- the k^{th} *central moment* of random variable X is

$$\mu_X^{(k)} := \mathbb{E}[(X - \mathbb{E}[X])^k],$$

  if it exists.

Remark

Chebyshev's inequality is most useful when we only have information about the *second moment* of our random variable X.

Higher Moments

To obtain even more information of a random variable, useful to see more of its moments:

- the *k^{th} moment* of random variable X is $\mathbb{E}[X^k]$.
- the *k^{th} central moment* of random variable X is

$$\mu_X^{(k)} := \mathbb{E}[(X - \mathbb{E}[X])^k],$$

  if it exists.

Remark

Chebyshev's inequality is most useful when we only have information about the *second moment* of our random variable X.

Practice problem: Can you generalize Chebyshev's inequality to k^{th} order moments?

Sums of Independent Random Variables

Often times in analysis of algorithms we deal with random variables which are sums of independent random variables (Distinct Elements, hashing, balls & bins, etc).

Sums of Independent Random Variables

Often times in analysis of algorithms we deal with random variables which are sums of independent random variables (Distinct Elements, hashing, balls & bins, etc).

Can we use this information to get better tail inequalities?

Sums of Independent Random Variables

Often times in analysis of algorithms we deal with random variables which are sums of independent random variables (Distinct Elements, hashing, balls & bins, etc).

Can we use this information to get better tail inequalities?

Law of large numbers: average of *independent, identically distributed variables* is *approximately* the *expectation* of the random variables. That is, if each X_i is an independent copy of random variable X

$$\frac{1}{n} \cdot \sum_{i=1}^{n} X_i \approx \mathbb{E}[X]$$

Sums of Independent Random Variables

Often times in analysis of algorithms we deal with random variables which are sums of independent random variables (Distinct Elements, hashing, balls & bins, etc).

Can we use this information to get better tail inequalities?

Law of large numbers: average of *independent, identically distributed variables* is *approximately* the *expectation* of the random variables. That is, if each X_i is an independent copy of random variable X

$$\frac{1}{n} \cdot \sum_{i=1}^{n} X_i \approx \mathbb{E}[X]$$

Central Limit Theorem: if we let $Z_n = \sum_{i=1}^{n} X_i$, where X_i independent copy of X, the random variable

$$Y_n = \frac{Z_n - n \cdot \mathbb{E}[X]}{\sqrt{n \cdot \sigma(X)^2}} \to \mathcal{N}(0, 1)$$

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

[1] Also works for sums of random variables which are not identically distributed!

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

Simple Setting: we have n coin flips, each is head with probability p. So

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } X = \sum_{i=1}^{n} X_i.$$

[1] Also works for sums of random variables which are not identically distributed!

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

Simple Setting: we have n coin flips, each is head with probability p. So

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } X = \sum_{i=1}^{n} X_i.$$

- Expected # heads: $n \cdot p$

[1] Also works for sums of random variables which are not identically distributed!

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

Simple Setting: we have n coin flips, each is head with probability p. So

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } X = \sum_{i=1}^{n} X_i.$$

- Expected # heads: $n \cdot p$
- To bound upper tail, need to compute:

$$\Pr[X \geq k] \leq \sum_{i \geq k} \binom{n}{i} p^i (1-p)^{n-i}$$

[1] Also works for sums of random variables which are not identically distributed!

Chernoff Bounds

Chernoff bounds give us quantitative estimates of the probability that X is far from $\mathbb{E}[X]$ for large enough values of n, when $X = X_1 + \cdots + X_n$.[1]

Simple Setting: we have n coin flips, each is head with probability p. So

$$X_i = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{otherwise} \end{cases} \qquad \text{and } X = \sum_{i=1}^{n} X_i.$$

- Expected # heads: $n \cdot p$
- To bound upper tail, need to compute:

$$\Pr[X \geq k] \leq \sum_{i \geq k} \binom{n}{i} p^i (1-p)^{n-i}$$

- Not easy to work with, hard to generalize

[1] Also works for sums of random variables which are not identically distributed!

Chernoff Bounds

Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } t > 0.$$

Chernoff Bounds

Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } t > 0.$$

What do we gain by doing this?

Chernoff Bounds

Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } t > 0.$$

What do we gain by doing this?

- The *moment generating function*

$$M_X(t) := \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{i \geq 0} \frac{t^i}{i!} \cdot X^i\right] = \sum_{i \geq 0} \frac{t^i}{i!} \cdot \mathbb{E}\left[X^i\right]$$

  contains information about all moments!

Chernoff Bounds

Generic Chernoff Bounds: apply Markov in the following way:

$$\Pr[X \geq a] = \Pr[e^{tX} \geq e^{ta}] \leq \mathbb{E}[e^{tX}]/e^{ta}, \quad \text{for any } t > 0.$$

What do we gain by doing this?

- The *moment generating function*

$$M_X(t) := \mathbb{E}[e^{tX}] = \mathbb{E}\left[\sum_{i \geq 0} \frac{t^i}{i!} \cdot X^i\right] = \sum_{i \geq 0} \frac{t^i}{i!} \cdot \mathbb{E}\left[X^i\right]$$

  contains information about all moments!
- If $X = X_1 + X_2$, where X_1, X_2 are independent, note that

$$\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX_1} e^{tX_2}] = \mathbb{E}[e^{tX_1}] \cdot \mathbb{E}[e^{tX_2}]$$

Chernoff Bounds for Bounded Variables

Example (Heterogeneous Coin Flips)

Let $X_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{otherwise} \end{cases}$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$

1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$

Chernoff Bounds for Bounded Variables

Example (Heterogeneous Coin Flips)

Let $X_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{otherwise} \end{cases}$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$

1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$
2. for $0 < \delta < 1$, $\Pr[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3}$

Chernoff Bounds for Bounded Variables

Example (Heterogeneous Coin Flips)

Let $X_i = \begin{cases} 1, & \text{with probability } p_i \\ 0, & \text{otherwise} \end{cases}$, $X = \sum_{i=1}^{n} X_i$ and $\mu = \mathbb{E}[X]$

1. for $\delta > 0$, $\Pr[X \geq (1+\delta)\mu] \leq \left[\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right]^{\mu}$
2. for $0 < \delta < 1$, $\Pr[X \geq (1+\delta)\mu] \leq e^{-\delta^2\mu/3}$
3. for $R \geq 6\mu$, $\Pr[X \geq R] \leq 2^{-R}$

Chernoff Bounds for Bounded Variables

What about the lower tail?

[2] See [Motwani & Raghavan 2007, Theorem 4.2] or [Mitzenmacher & Upfal, Theorem 4.5]

Chernoff Bounds for Bounded Variables

What about the lower tail?

Similar proof, by setting $t < 0$.[2]

[2] See [Motwani & Raghavan 2007, Theorem 4.2] or [Mitzenmacher & Upfal, Theorem 4.5]

Chernoff Bounds for Bounded Variables

What about the lower tail?

Similar proof, by setting $t < 0$.[2]

Theorem (Heterogeneous Coin Flips - lower tail)

1. $\Pr[X \leq (1-\delta)\cdot\mu] \leq \left[\dfrac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right]^{\mu}$
2. *if* $0 < \delta < 1$ *then* $\Pr[X \leq (1-\delta)\cdot\mu] \leq e^{-\mu\delta^2/2}$

[2] See [Motwani & Raghavan 2007, Theorem 4.2] or [Mitzenmacher & Upfal, Theorem 4.5]

Hoeffding's generalization

What if the variables X_i took values in $[a_i, b_i]$?

Hoeffding's generalization

What if the variables X_i took values in $[a_i, b_i]$?

Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^n X_i$. Then

$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^n (b_i - a_i)^2}\right)$$

Hoeffding's generalization

What if the variables X_i took values in $[a_i, b_i]$?

Theorem (Hoeffding's Inequality)

Let X_i be independent random variables, taking values in $[a_i, b_i]$, $X = \sum_{i=1}^{n} X_i$. Then

$$\Pr[|X - \mathbb{E}[X]| \geq \ell] \leq 2 \cdot \exp\left(-\frac{2\ell^2}{\sum_{i=1}^{n}(b_i - a_i)^2}\right)$$

Proof uses *Hoeffding's lemma:* $\mathbb{E}[e^{t(X_i - \mathbb{E}[X_i])}] \leq \exp\left(\frac{t^2(b_i - a_i)^2}{8}\right)$

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected # heads is $n/2$. Chernoff-Hoeffding implies:

$$\Pr[|\#\text{ heads } - \mu| \geq \delta\mu] \leq 2\exp(-\mu\delta^2/3) = 2\exp(-n\delta^2/6)$$

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected # heads is $n/2$. Chernoff-Hoeffding implies:

$$\Pr[|\#\text{ heads } - \mu| \geq \delta\mu] \leq 2\exp(-\mu\delta^2/3) = 2\exp(-n\delta^2/6)$$

- Setting $\delta = \sqrt{60/n}$, probability above is $\leq 2e^{-10}$. Thus

$$\Pr[|\#\text{ heads } - n/2| \geq \sqrt{15 \cdot n}] \leq 2e^{-10}.$$

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected # heads is $n/2$. Chernoff-Hoeffding implies:

$$\Pr[|\#\text{ heads } - \mu| \geq \delta\mu] \leq 2\exp(-\mu\delta^2/3) = 2\exp(-n\delta^2/6)$$

- Setting $\delta = \sqrt{60/n}$, probability above is $\leq 2e^{-10}$. Thus

$$\Pr[|\#\text{ heads } - n/2| \geq \sqrt{15 \cdot n}] \leq 2e^{-10}.$$

- With high probability, # heads is within $O(\sqrt{n})$ of the expected value (this comes up in many places). **Practice problem:** prove that with constant probability that $|\#\text{ heads } - n/2| = \Omega(\sqrt{n})$.

Remarks

- In coin flips example from beginning of lecture, by flipping n independent fair coins, expected # heads is $n/2$. Chernoff-Hoeffding implies:

$$\Pr[|\#\text{ heads } - \mu| \geq \delta\mu] \leq 2\exp(-\mu\delta^2/3) = 2\exp(-n\delta^2/6)$$

- Setting $\delta = \sqrt{60/n}$, probability above is $\leq 2e^{-10}$. Thus

$$\Pr[|\#\text{ heads } - n/2| \geq \sqrt{15 \cdot n}] \leq 2e^{-10}.$$

- With high probability, # heads is within $O(\sqrt{n})$ of the expected value (this comes up in many places). **Practice problem:** prove that with constant probability that $|\#\text{ heads } - n/2| = \Omega(\sqrt{n})$.
- From previous slides:

$$\text{Markov: } \Pr[\#\text{ heads } \geq 3n/4] \leq 2/3$$

$$\text{Chebyshev: } \Pr[\#\text{ heads } \geq 3n/4] \leq 4/n.$$

$$\text{Chernoff: } \Pr[\#\text{ heads } \geq 3n/4] \leq e^{-n/24}.$$

Remarks

- It is often easier to compute moments by computing the moment generating functions

Remarks

- It is often easier to compute moments by computing the moment generating functions
- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)

Remarks

- It is often easier to compute moments by computing the moment generating functions
- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)
- Chernoff-Hoeffding bounds also hold for *negatively correlated* variables, because all we need is

$$\mathbb{E}[e^{t(X+Y)}] \leq \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$$

  This observation is very useful in many applications (also great source of final projects!)

Remarks

- It is often easier to compute moments by computing the moment generating functions
- Why do we want to compute moments? See Sum-of-Squares and pseudo-distributions references in course webpage. These methods give very powerful tools to solve many challenging problems! (great final project topic!)
- Chernoff-Hoeffding bounds also hold for *negatively correlated* variables, because all we need is

$$\mathbb{E}[e^{t(X+Y)}] \leq \mathbb{E}[e^{tX}] \cdot \mathbb{E}[e^{tY}]$$

  This observation is very useful in many applications (also great source of final projects!)
- For instance: two edges appear in a random spanning tree is a negatively correlated event, thus Chernoff bounds are useful to analyze random spanning trees.

Acknowledgement

- Lecture based largely on Lap Chi's notes and [Motwani & Raghavan 2007, Chapters 3 and 4].
- See Lap Chi's notes at `https://cs.uwaterloo.ca/~lapchi/cs466/notes/L02.pdf`

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms

Mitzenmacher, Michael, and Eli Upfal (2017)
Probability and computing: Randomization and probabilistic techniques in algorithms and data analysis.
Cambridge university press, 2017.