Lecture 1: Amortized Analysis

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May 9, 2023

Overview

- Introduction
  - Why amortized analysis?
  - Types of amortized analyses
- Examples of Data Structures Using Amortized Analysis
  - Aggregate Analysis
  - Accounting Method
  - Potential Method
- Acknowledgements

Why Amortized Analysis?

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Worst or average-case complexity of data structures

Data Structure	search	insertion	deletion
Doubly-Linked List	$O(n)$	$O(1)$	$O(n)$
Ordered Array	$O(\log n)$	$O(n)$	$O(n)$
Hash Tables[a]	$O(1)$	$O(1)$	$O(1)$
Balanced Binary Search Trees[b]	$O(\log n)$	$O(\log n)$	$O(\log n)$

[a] Average-case, although worst-case search time is $\Theta(n)$

[b] Also average-case. Worst-case complexity is $O(\mathit{height})$ of the tree, which can be $\Theta(n)$.

Why Amortized Analysis?

In **amortized analysis**, one averages the *total time* required to perform a sequence of data-structure operations over *all operations performed*.

Upshot of amortized analysis: worst-case cost *per query* may be high for one particular query, so long as overall average cost per query is small in the end!

Remark

Amortized analysis is a *worst-case* analysis. That is, it measures the average performance of each operation in the worst case.

Types of amortized analyses

Three common types of amortized analyses:

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2. **Accounting Method:** assign certain *charge* to each operation (independent of the actual cost of the operation). If operation is cheaper than the charge, then build up credit to use later.
3. **Potential Method:** one comes up with *potential energy* of a data structure, which maps each state of entire data-structure to a real number (its "potential"). Differs from accounting method because we assign credit to the data structure as a whole, instead of assigning credit to each operation.

One simple problem - several analyses

- **Input:** A binary counter C initially set to zero
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$n = 8$ $\log n = 3$

0111

1000

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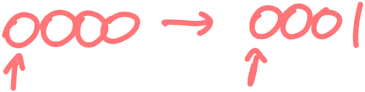
- Is the above analysis tight?



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- How many times will we "flip" the k^{th} bit?

i.e. after 2^{k-1} operations/increments

flip k^{th} bit in intervals of 2^{k-1} increments

$\Rightarrow \left\lfloor n / 2^{k-1} \right\rfloor$ flips of k^{th} bit

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- Is the above analysis tight?
- How many times will we "flip" the k^{th} bit?
- Putting it all together, we get:

$$\sum_{k=0}^{\lceil \log n \rceil} \underbrace{\lfloor n/2^k \rfloor}_{\#\text{ times } k^{th} \text{ bit gets flipped}} < \sum_{k \geq 0} n/2^k = 2n$$

$\Rightarrow$ amortized cost per op. $\Theta(1)$ (total $\Theta(n)$)

Accounting method

- suppose that the *actual cost* of each operation of our algorithm is c_i (which may be hard to track)

$$\sum_{i=1}^{n} c_i$$

$c_i :=$ cost of i^{th} increment (# bit flips)

$c_1 = 1$ $\quad$ $0000 \mapsto 0001$

$c_2 = 2$ $\quad$ $0001 \mapsto 0010$

$c_3 = 1$ $\quad$ $0010 \mapsto 0011$

$c_4 = 3$

Accounting method

- suppose that the *actual cost* of each operation of an algorithm is c_i

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- In the accounting method, at each step of the algorithm, we assign *charges* γ_i to each operation such that

$$\sum_{i=1}^{\ell} \gamma_i \geq \sum_{i=1}^{\ell} c_i$$

  for any $\ell \geq 1$

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- If we manage to do the above, then

$$\text{Total cost} \leq \text{Total charged}$$

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$0000 \longmapsto 0001$

$\uparrow$

$\gamma_1 = 2$

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- If we flip k bits during an increment, we have already charged $k - 1$ of those bit flips to earlier bit flips.

Why?

$k = 3$

$011 \mapsto 100$

change k → clear $k-1$ bits
set 1 bit

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- So, instead of *paying* for k bit flips in this increment, we *charge* at most 2:
  - one for setting a bit to 1, *actual cost*
  - and the other is the charge to "clear this bit" *clearing charge*

$\gamma_i = 2$

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- Total cost $\leq$ Total Charged $= 2 \times n$

Example of the accounting method

$\gamma_1 = 2$

(1 ← setting one bit to 1)

(1 ← clearing this bit (maybe done later))

$\sum \gamma_i = 2 \times n = 2n$

Formal Analysis of the accounting method

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$$\gamma_i = c_i + \Phi_i - \Phi_{i-1}$$

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- We have:

$$\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} (c_i + \Phi_i - \Phi_{i-1}) = \Phi_n - \Phi_0 + \sum_{i=1}^{n} c_i$$

total amortized cost

telescoping

actual cost

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≥ 0

- So if $\Phi_k - \Phi_0 \geq 0$ for all $k \geq 0$ (*valid potential function*) the total amortized cost is an *upper bound* on total cost.

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- Potential:

$$\Phi_i = \text{ number of bits with value 1 at step } i$$

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- $\Phi_0 = 0$ and $\Phi_i = \underbrace{\# \text{ of 1 bits of } i}_{\geq 0} \geq 0$ (valid potential function)

$\forall k \quad \phi_k \geq \phi_0 = 0 \quad$ valid

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- What is the amortized cost of the i^{th} operation:
  - $c_i = (\# \text{ bits } 0 \to 1) + (\# \text{ bits } 1 \to 0)$ *cost*

set clearing

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- Amortized cost:

$$\gamma_i = c_i + \Phi_i - \Phi_{i-1} = 2 \times (\#\text{ bits } 0 \to 1)$$

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- Since each increment *only changes 1 bit from 0 to 1* each amortized cost is 2.

Example of the potential method

$$\sum c_i \le \sum c_i + \underbrace{\Phi_n - \Phi_0}_{\text{valid}}$$

$$= \sum (c_i + \Phi_i - \Phi_{i-1})$$

$$= \sum \gamma_i = 2 \cdot n$$

Discussion of the potential method

Acknowledgements

- Lecture largely based on Jeff Erickson's notes (with exercises!)

  http://jeffe.cs.illinois.edu/teaching/algorithms/notes/09-amortize.pdf

- More exercises and another example using all methods can also be found at the [CLRS] book, chapter 17. (see useful resources page)