Lecture 7: Multivariate Polynomial Division Algorithm & Monomial Ideals

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Overview

- Two Familiar Division Algorithms
- Generalization: Multivariate Multipolynomial Division
- Issues with the division algorithm
- Monomial Ideals & Dickson's Lemma
- Conclusion
- Acknowledgements

Division with remainder over $\mathbb{F}[x]$

- **Input:** two elements $a, b \in \mathbb{F}[x]$, with b non-zero
- **Output:** $q, r \in \mathbb{F}[x]$ such that $\deg(r) < \deg(b)$ and $a = q \cdot b + r$

$\mathbb{F}[x]$ is Euclidean domain

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- Start with $r = a$, $q = 0$
- While $\deg(r) \geq \deg(b)$:
  - ~~$q \leftarrow q + x^{\deg(r)-\deg(b)}$~~
  - $r \leftarrow r - x^{\deg(r)-\deg(b)} \cdot \frac{LC(r)}{LC(b)} \cdot b$ ← kill the biggest monomial power

leading coefficients match

$$\deg(r') \leq \deg(r) - 1$$

$$q \leftarrow q + \frac{LC(r)}{LC(b)} \cdot x^{\deg(r)-\deg(b)}$$

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- Start with $r = a$, $q = 0$
- While $\deg(r) \geq \deg(b)$:
  - $q \leftarrow q + x^{\deg(r) - \deg(b)}$
  - $r \leftarrow r - x^{\deg(r) - \deg(b)} \cdot \frac{LC(r)}{LC(b)} \cdot b$
- Analysis: we will perform at most $\deg(a) - \deg(b) + 1$ subtractions to r. Total time $(\deg(a) - \deg(b) + 1)(\deg(b) + 1)$.

Example

$x = 7/8$

- $a(x) = x^3 + 2x^2 + x + 1,\ b(x) = 2x + 1$

$q : \frac{1}{2}x^2 + \frac{3}{4}x + \frac{1}{8}$

$2x + 1 \;\big|\; x^3 + 2x^2 + x + 1$

$x^3 + \frac{1}{2}x^2$

$\frac{3}{2}x^2 + x + 1$

$\frac{3x^2}{2} + \frac{3x}{4}$

$\frac{x}{4} + 1$

$\frac{x}{4} + \frac{1}{8}$

$7/8 \rightarrow 0$

Solving Linear System - Gaussian Elimination

- **Input:** matrix $A \in \mathbb{F}^{n \times d}$, vector $b \in \mathbb{F}^{n}$
- **Output:** Is there a solution $y \in \mathbb{F}^{d}$ to $Ay = b$?

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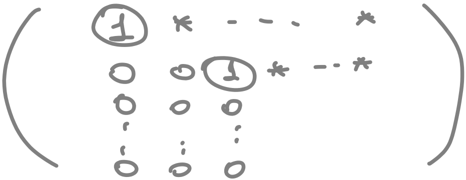
  1. Put $C = \begin{pmatrix} A & b \end{pmatrix}$ in reduced row-echelon form *we will focus on this*



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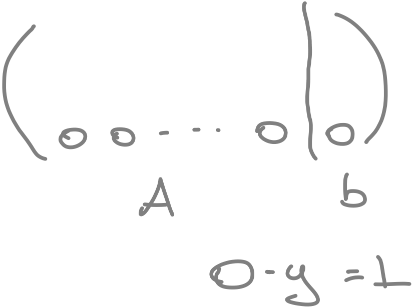
  2. From bottom-up along rows of A, if the equation has a solution then set it properly



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  3. So long as there are no inconsistencies, we found a solution

Example

- $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$

Why Multivariate Multipolynomial Division?

- From last lecture, many algorithmic problems we really would like to solve:
  1. ideal membership problem
  2. solving polynomial equations
  3. implicitization problem
  4. finding irreducible components of algebraic set
  5. among others...

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- Implicit in the seminal works of Hilbert and Gordan from 1890s!
- Complexity analyzed by Buchberger in 1960s!

Monomial Ordering

- In division algorithm over $\mathbb{F}[x]$, implicitly assumed $x \leq x^2 \leq x^3 \leq \cdots$ and that constants were "smaller than" any power of x

$1 < x < x^2 < x^3 < \cdots$

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Can (sometimes should) do it in several ways!

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- Even to write a polynomial in a "humanly consistent way" we assume a monomial order (i.e., the ones we write first)
- Example: given two monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathbb{F}[x_1, \ldots, x_n]$, we say

$$\mathbf{x}^{\mathbf{a}} \succeq \mathbf{x}^{\mathbf{b}} \quad \text{if} \quad \mathbf{a} \geq \mathbf{b} \quad \text{in lexicographic order over } \mathbb{N}^n$$

$x > y$ $\qquad$ $x^2 > xy^3$

$x > y^2$ $\qquad$ (a, b) $\quad$ (c, d)

$a > c$ or $a = c$

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- In general a good monomial order has:
  1. *Total order*: any two elements can be compared
  2. *Transitive*: $\mathbf{x}^{\mathbf{a}} \succeq \mathbf{x}^{\mathbf{b}}$ and $\mathbf{x}^{\mathbf{b}} \succeq \mathbf{x}^{\mathbf{c}}$ then $\mathbf{x}^{\mathbf{a}} \succeq \mathbf{x}^{\mathbf{c}}$
  3. *Well-behaved under multiplication:* $\mathbf{x}^{\mathbf{a}} \succeq \mathbf{x}^{b} b \Rightarrow \mathbf{x}^{\mathbf{a}+\mathbf{c}} \succeq \mathbf{x}^{\mathbf{b}+\mathbf{c}}$
  4. *Well-ordering*: every non-empty subset has a *smallest element*

Leading Terms, Monomials, Coefficients

$\bar{x}^{\bar{e}} = \prod_{i=1}^{n} x_i^{e_i}$ $\bar{x} = (x_1, \dots, x_n)$

- Now we are ready to define special terms of polynomials

$$f(\mathbf{x}) = \sum_{\alpha} f_\alpha \cdot \mathbf{x}^\alpha$$

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$$\text{supp}(f) := \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}$$

all monomials α
where coeff. $f_\alpha \neq 0$

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- The *support* of f

$$\mathrm{supp}(f) := \{\alpha \in \mathbb{N}^n \mid f_\alpha \neq 0\}$$

- The *multidegree* of f is the maximum monomial in the support of f according to $\succeq$. Termed $\mathrm{multideg}(f)$. (1, 0) (0, 3)
- The *leading monomial* of f is $LM(f) := \mathbf{x}^{\mathrm{multideg}(f)}$ x y^3
- The *leading coefficient* of f is $LC(f) := f_{\mathrm{multideg}(f)}$ 3 1
- The *leading term* of f is $LC(f) \cdot LM(f)$. $3x$ y^3

lex order $3x + y^2 + y^3$

graded lex

A Division Algorithm - a first attempt

- **Input:** polynomials $G, F_1, \ldots, F_s \in \mathbb{F}[\mathbf{x}]$ and a monomial order $\succeq$
- **Output:** $Q_1, \ldots, Q_s, R \in \mathbb{F}[\mathbf{x}]$ such that

$$G = F_1 \cdot Q_1 + \cdots + F_s \cdot Q_s + R$$

where $\text{multideg}(R) < \text{multideg}(F_i)$ for $i \in [s]$

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- Example 1: $G = xy^2 + 1$, $F_1 = xy + 1$ and $F_2 = y + 1$

lex order

$Q_1 : y$
$Q_2 : -1$

$F_1 = xy + 1$
$F_2 = y + 1$

$xy^2 + 1$
$xy^2 + y$
$-y + 1$
$-y - 1$
2 → 0

$r = 2$

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- Quotients are not unique: $F_1 = y+1$ $F_2 = xy+1$

$$xy^2 + 1 = xy \cdot (y + 1) + (-1) \cdot (xy + 1) + 2$$

Division Algorithm - Subtlety

- The following subtlety comes because we have more than one variable
- Example 2: $G = x^2y + xy^2 + y^2$, $F_1 = xy - 1$ and $F_2 = y^2 - 1$ with lex order

$Q_1 : x + y$

$Q_2 : 1$

$F_1 = xy - 1$

$F_2 = y^2 - 1$

$$
\begin{array}{l}
x^2y + xy^2 + y^2 \\
\underline{x^2y - x} \\
xy^2 + x + y^2 \\
\underline{xy^2 - y} \\
\underline{x + y^2 + y} \\
y^2 + y \\
\underline{y^2 - 1} \\
y + 1 \rightarrow 1 \rightarrow 0
\end{array}
$$

$r = x + y + 1$

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- Thus we have

$$x^2y + xy^2 + y^2 = (x + y) \cdot (xy - 1) + 1 \cdot (y^2 - 1) + (x + y + 1)$$

- So, instead of requiring that the leading term of remainder be smaller than leading term of divisors, better to require that *no monomial* of R is divisible by *any leading monomial* of the F_i's

A Division Algorithm - second attempt

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no monomial of R be divisible by *any leading term* of the F_i's.
Furthermore if $F_i Q_i \neq 0$, we also want:

$$LM(G) \succeq LM(F_i Q_i)$$

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- Algorithm:
  1. While $LM(G)$ is divisible by some $LM(F_i)$, divide appropriately (respecting the order preference of F_i's)
  2. If no $LM(F_i) \mid LM(G)$, add $LT(G)$ to the remainder and go back to step 1

A Division Algorithm - second attempt

- **Input:** polynomials $G, F_1, \ldots, F_s \in \mathbb{F}[\mathbf{x}]$ and monomial ordering
- **Output:** $Q_1, \ldots, Q_s, R \in \mathbb{F}[\mathbf{x}]$ such that

$$G = F_1 \cdot Q_1 + \cdots + F_s \cdot Q_s + R$$

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- The algorithm above always terminates.

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  2. If no $LM(F_i) \mid LM(G)$, add $LT(G)$ to the remainder and go back to step 1
- The algorithm above always terminates.
- Proof is by well-ordering principle of the monomial order and fact that each step of division algorithm decreases leading term of G.

$\alpha(1) > \alpha(2) > \alpha(3) > \cdots \longrightarrow$ must be finite

Pseudocode

How does this generalize the two previous algorithms?

- Note that for univariate polynomials, the division algorithm works in the same way, if we consider the leading term of G one at a time

[1] This is more appropriate when checking if a linear form is in the span of a set of other linear forms

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- Note that for univariate polynomials, the division algorithm works in the same way, if we consider the leading term of G one at a time
- For row-echelon form, note that it is exactly the division algorithm when the polynomials are linear[1]

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Does it have the properties we want?

- What properties would we want from a division algorithm?
  1. remainder should be *uniquely determined*
  2. ordering shouldn't really matter (especially since we are trying to use it to solve ideal membership problem)
  3. univariate division algorithm solves ideal membership problem - so our division algorithm should also solve it

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- Example 3: $G = x^2y + xy^2 + y^2$, $F_1 = y^2 - 1$ and $F_2 = xy - 1$ with lex order

same as example 2 with order reversed

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$$\begin{array}{l} x^2y + xy^2 + y^2 \\ \underline{x^2y - x} \\ xy^2 + x + y^2 \\ \underline{xy^2 - x} \\ 2x + y^2 \end{array}$$

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$r = 2x + 1$

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- Example 3: $G = x^2y + xy^2 + y^2$, $F_1 = y^2 - 1$ and $F_2 = xy - 1$ with lex order same as example 2 with order reversed
- Note that remainder here is $2x + 1$, which is different from remainder in example 2: $(x + y + 1)$

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- Our division algorithm only gives *sufficient* condition for ideal membership problem: if G has zero remainder when divided by $(F_1, \ldots, F_s)$ then we know $G \in (F_1, \ldots, F_s)$

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- Example 4: $G = xy^2 - x$, $F_1 = xy - 1$ and $F_2 = y^2 - 1$

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- Example 4: $G = xy^2 - x$, $F_1 = xy - 1$ and $F_2 = y^2 - 1$
- The "fix" for this division algorithm is to find a *good basis* for the ideal generated by $F_1, \ldots, F_s$ - the so-called Gröbner basis

Description of Ideals

- In the definition of algebraic sets, we used any family of polynomials $\mathcal{F}$ to define an algebraic set (or the ideal $I_{\mathcal{F}}$).

Question

Does every ideal of $\mathbb{F}[x_1, \ldots, x_n]$ have a finite description?

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- Today we will address this question for *monomial ideals*. This will be done by Dickson's lemma[2]
- A monomial ideal is any ideal generated by a family $\mathcal{F}$ of monomials (not necessarily a finite number of them)

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- In the definition of algebraic sets, we used any family of polynomials $\mathcal{F}$ to define an algebraic set (or the ideal $I_{\mathcal{F}}$).

Question

Does every ideal of $\mathbb{F}[x_1, \ldots, x_n]$ have a finite description?

- Today we will address this question for *monomial ideals*. This will be done by Dickson's lemma[2]
- A monomial ideal is any ideal generated by a family $\mathcal{F}$ of monomials (not necessarily a finite number of them)

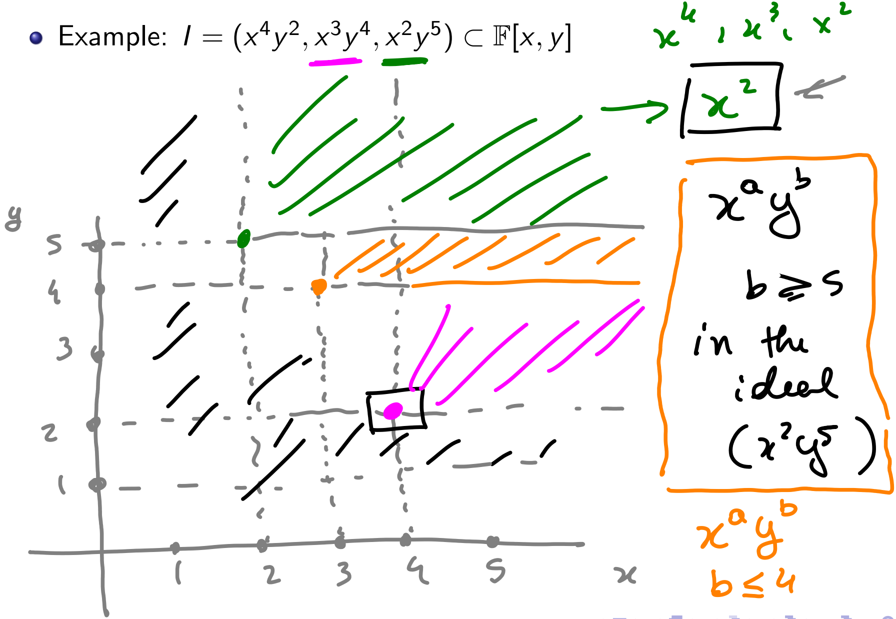
Theorem (Dickson's lemma)

Let $I = (\mathbf{x}^\alpha \mid \alpha \in \mathcal{F}) \subset \mathbb{F}[x_1, \ldots, x_n]$ be a monomial ideal. Then I can be written as $I = (\mathbf{x}^{\alpha(1)}, \ldots, \mathbf{x}^{\alpha(s)})$, where $\alpha(1), \ldots, \alpha(s) \in \mathcal{F}$

[2] Which was in fact first proved by Gordan.

Dickson's Lemma - picture & example

- Example: $I = (x^4y^2, x^3y^4, x^2y^5) \subset \mathbb{F}[x, y]$



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ordering $\Longleftrightarrow$ being divisible

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  3. Let $J \subseteq \mathbb{F}[x_1, \ldots, x_n]$ be the monomial ideal generated by $\mathbf{x}^\alpha$ such that $\mathbf{x}^\alpha \cdot y^m \in I$ for some $m \geq 0$.

J "projection" of I over $\mathbb{F}[x_1, \ldots, x_n]$

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  4. J is finitely generated, say $J = (\mathbf{x}^{\alpha(1)}, \cdots, \mathbf{x}^{\alpha(s)})$

hypothesis

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  5. Let $m_i \in \mathbb{N}$ be smallest integer such that $\mathbf{x}^{\alpha(i)} \cdot y^{m_i} \in I$, and let $N := \max m_i$. And let $I_N := (\mathbf{x}^{\alpha(1)} \cdot y^{m_1}, \ldots, \mathbf{x}^{\alpha(s)} \cdot y^{m_s})$

$I_N \subset I$

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  6. Any $\mathbf{x}^{\beta} y^m$ in I such that $m \geq N$ is in I_N.

$x^{\beta} y^m$ divisible by $x^{\beta} y^N$

and projection $x^{\beta} \in (\bar{x}^{\alpha(1)}, \ldots, \bar{x}^{\alpha(1)})$

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by hypothesis

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  8. Let $I_\ell := (\mathbf{x}^{\alpha_\ell(1)} \cdot y^\ell, \ldots, \mathbf{x}^{\alpha_\ell(s_\ell)} \cdot y^\ell)$
  9. Show that $I = I_0 + I_1 + \cdots + I_N$

Proof of Dickson's lemma

$(I_N + I_{N-1} + \cdots + I_1 \subseteq I)$

$x^\beta y^m \in I$

if $m \geq n \Rightarrow x^\beta y^m \in I_N$

if $m < n \Rightarrow x^\beta \in J_m \Rightarrow$

$x^\beta y^m \in I_m$

$\Rightarrow I \subset I_1 + \cdots + I_N$

$\Rightarrow I = I_1 + \cdots + I_N$

Consequences of Dickson's lemma

- Dickson's lemma helps us decide if a monomial relation is a proper *monomial ordering*

Corollary (Monomial Order Criterion)

If $>$ *is a relation on* $\mathbb{N}^n$ *satsifying*

1. $>$ *is a total ordering on* $\mathbb{N}^n$
2. $\alpha > \beta$ *and* $\gamma \in \mathbb{N}^n$ *then* $\alpha + \gamma > \beta + \gamma$

Then $>$ *is a well-ordering if, and only if,* $\alpha \geq 0$ *for all* $\alpha \in \mathbb{N}^n$*.*

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  - elimination ordering (lex order)
  - graded rev-lex order used in most ideal membership tasks

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- As we will see later in the course, this is great as different monomial orderings are used for different purposes.
  - elimination ordering
  - graded rev-lex order used in most ideal membership tasks
- From the set of bases for a monomial ideal, there is one which is better than others:

A *minimal basis* of a monomial ideal is one where none of the generators is divisible by another generator.

Conclusion

- Today we learned about the division algorithm and Dickson's lemma
- Division algorithm generalizes univariate division algorithm and Gaussian elimination
- Division algorithm is not great - we will fix that by finding a good basis
- Dickson's lemma shows that monomial ideals are finitely generated
- Can use it to have easy criterion for checking monomial orderings
- Will use this lemma to prove Hilbert Basis Theorem

Acknowledgement

- Lecture based entirely on the book by CLO: Ideals, varieties and algorithms (see course webpage for a copy - or get online version through UW library)