Lecture 5: Barriers to Lower Bound Techniques & Algebraic Natural Proofs

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Overview

- Lower bound approaches - Rank Methods
- Barriers to Rank Methods
- Algebraic Natural Proofs & Succinct PIT
- Conclusion
- Acknowledgements

Lower Bound Approach

1. Define class of simple polynomials $\mathcal{S}$
2. *Normal form:* every circuit from circuit class $\mathcal{C}$ can be expressed as small sum of simple polynomials in $\mathcal{S}$

$$\Phi \in \mathcal{C}(s) \implies \Phi = \underline{f_1 + \cdots + f_s}$$

$$f_i \in \mathcal{S}$$

Can think of class $\mathcal{C}$ as $\mathcal{S}$-complexity

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$$\mu(f+g) \leq \mu(f) + \mu(g)$$

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- If $\mu(f) \leq U$ for all $f \in \mathcal{S}$ $\mu(S) \leq U$

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- If $\mu(f) \leq U$ for all $f \in \mathcal{S}$
- By sub-additivity $\mu(q) \leq s \cdot U$ for any $q \in \mathcal{C}$ which can be written as

$$q = f_1 + f_2 + \cdots + f_s, \quad f_i \in \mathcal{S}$$

$$\mu(q) = \mu(f_1 + \cdots + f_s) \leq \mu(f_1) + \cdots + \mu(f_s)$$
$$\leq s \cdot U$$

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$$q = f_1 + f_2 + \cdots + f_s, \quad f_i \in \mathcal{S}$$
- $\mu(p) \geq L$ and p can be computed by size s in $\mathcal{C} \Rightarrow s \cdot U \geq L$

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hom-$\Sigma\Pi\Sigma$ hom-$\Sigma\wedge^d\Sigma\Pi^2$

Common aspects of complexity measures - rank methods

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2. Dimension of span of: partial derivatives, shifted partial derivatives
3. Can be cast as *ranks of special matrices*:

$$f+g \longmapsto L(f+g) = L(f) + L(g)$$

$$L : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}^{m \times m} \qquad \text{linear map}$$

$$f \longmapsto L(f) = L(\mathrm{coeff}(f))$$

$$\mu : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N} \qquad \mu(f) = \mathrm{rank}(L(f))$$

$$f(x_1, x_2, x_3, x_4) = E_{4,3} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4$$

	x_1x_2	x_1x_3	x_1x_4	x_2x_3	x_2x_4	x_3x_4
x_1	0	0	0	1	1	1
x_2	0	1	1	0	0	1
x_3						
x_4						

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4. Sub-additivity comes from sub-additivity of rank

$$\mathrm{rank}(A+B) \leq \mathrm{rank}(A) + \mathrm{rank}(B)$$

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5. Examples:
   - dimension of partial derivatives $\to$ rank of partial derivative matrix
   - dimension of shifted paritals $\to$ same as above
   - Flattenings used in tensor rank lower bounds $\to$ flattening is such a matrix map!

Partial derivatives method as rank method

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What is a barrier?

- Given a class of simple polynomials $\mathcal{S}$, let $c_{\mathcal{S}}(p)$ be the *$\mathcal{S}$-complexity* of polynomial p - that is, the min s such that

$$p = f_1 + \ldots , f_s, \quad f_i \in \mathcal{S}$$

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- Assume that $\mathcal{S}$ is *complete* – that is, any polynomial in $\mathbb{F}[x_1, \ldots, x_n]$ can be computed by the span of polynomials in $\mathcal{S}$

$S = \{ \prod_{i=1}^{d} \ell_i \mid$ linear forms of degree $d \}$

$x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ where $e_1 + \cdots + e_n = d$

$\in S$

know there are hard poly $\mathbb{F}[x_1, \ldots, x_n]_d$

Barrier: if $\mu(f)$ small $\forall f \in S$ then $\mu(p)$ is not too large for any polynomial in $\mathbb{F}[x_1, \ldots, x_n] = \mathrm{span}(S)$

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- Let $\underline{\Delta_{\mathcal{S}}}$ be set of all sub-additive measures over $\mathcal{S}$

all possible lower bound techniques (that we are considering)

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- $c_{\mathcal{S}} \in \Delta_{\mathcal{S}}$, but it is hard to understand

$$c_S(p+q) \le c_S(p) + c_S(q)$$

$$p = f_1 + \dots + f_r \qquad r = c_S(p)$$

$$q = g_1 + \dots + g_t \qquad t = c_S(q)$$

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$\Delta \leftarrow$ rank methods
(very easy to analyze)

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- A *barrier* for the subset $\boxed{\Delta}$ is a statement of the following kind:

  If $\underline{\mu \in \Delta}$ and $\underline{\mu(f)}$ is small for every $f \in \mathcal{S}$, then it is small for every $\underline{p \in \mathbb{F}[x_1, \ldots, x_n]}$

$\mu(p) / \mu(s)$ small (this ratio is our lower bound)

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- The above would rule out even *non-explicit* lower bounds!

Barriers to rank methods

- Let $\mathcal{S}$ be the class of powers of linear forms (Waring Rank)

$$S = \{ (a_1 x_1 + \cdots + a_n x_n)^d \mid (a_1, \ldots, a_n) \in \mathbb{F}^n \}$$

S is complete

write any monomial in span(S)

Barriers to rank methods

- Let $\mathcal{S}$ be the class of powers of linear forms ✓ (Waring Rank)
- A simple dimension count over $\boxed{\mathbb{F}[x_1, \ldots, x_n]_d}$ shows us that we must have polynomials requiring n^{d-1}

simple polynomials

$$n^d \approx \binom{n+d-1}{n-1} = \text{dim polys of deg } d$$

n vars

$$\sum_{\ell=1}^{t} \underbrace{(a_{\ell 1} x_1 + \cdots + a_{\ell n} x_n)}_{n \text{ degrees of freedom}}^{d}$$

$$\boxed{t \approx n^{d-1}}$$

$n \cdot t \approx n^d$

match degrees of freedom

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Practice problem

Barriers to rank methods

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Theorem ([Efremenko et al. 2018])

Rank methods cannot prove lower bounds better than $n^{d/2}$ for Waring Rank.

take easiest lower bd rank method
this is as good as any rank method
cannot give you non-trivial lower bds

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really weak

Theorem ([Efremenko et al. 2018])

Rank methods cannot prove lower bounds better than $n^{d/2}$ for Waring Rank.

- Note that this implies a barrier for depth-3 circuits as well!

S ← Waring rank

T ← hom. depth 3

S weaker than T

Barrier for Waring Rank - Symbolic Rank

- Going from generic rank to symbolic rank

$$L\left((a_1x_1+\cdots+a_nx_n)^d\right) \longrightarrow M(a_1,\ldots,a_n)$$

a_1^d

$\mathrm{rank}(M(a_1,\ldots,a_n)) \leq r$

$\forall (a_1,\ldots,a_n) \in \mathbb{F}^n$

measure is small for any element in S

symbolic matrix

$$M(y_1,\ldots,y_n) = \sum_{|\bar{e}|=d} M_{\bar{e}}\, \bar{y}^{\bar{e}}$$

polynomial matrix homogeneous of degree d

$M_{(d,0,\ldots,0)} \cdot y_1^d + \cdots$

$\mathrm{rank}_{\mathbb{F}(y)}(M(y)) \leq r$

$\det(M(y)) \neq 0$

$\Rightarrow M(\bar{a})$ invertible

Symbolic Rank to Small Decomposition

- Small symbolic rank $\Rightarrow$ small decomposition in field of fractions

$$M(\bar{y}) \qquad \mathrm{rank}_{\mathbb{F}(\bar{y})}\left(M(\bar{y})\right) \leq r$$

$$M(\bar{y}) = A(\bar{y}) \cdot B(\bar{y}) \qquad A \in \mathbb{F}(\bar{y})^{m \times r}, \quad B \in \mathbb{F}(\bar{y})^{r \times m}$$

$$M(\bar{y}) = \sum_{i=1}^{r} \frac{1}{g_i(\bar{y})} \cdot \underbrace{\vec{u}_i(\bar{y}) \cdot \vec{v}_i(\bar{y})^T}_{\text{rank-1 decompositions}}$$

$$g_i(\bar{y}) \in \mathbb{F}[\bar{y}] \qquad \vec{u}_i, \vec{v}_i \in \mathbb{F}[\bar{y}]^m$$

From field of fractions to polynomials

- From field of fractions decomposition, obtain small polynomial matrix decomposition

$$M(\bar{y}) = \sum_{i=1}^{r} \vec{u}_i(\bar{y}) \cdot \vec{v}_i(\bar{y})^T$$

polynomial vectors

Grouping elements based on degree

- Each rank-1 polynomial matrix can be broken down into pieces of degree $\leq d/2$

degree d

$$M(\bar{y}) = \sum_{i=1}^{r} \underbrace{\vec{u}_i(\bar{y})}\ \underbrace{\vec{v}_i(\bar{y})}^T$$

homogeneous polynomials

$$= \sum_{|\bar{e}|=d} M_{\bar{e}} \cdot \bar{y}^{\bar{e}}$$

$\mathbb{F}^{m \times m}$

$$\begin{pmatrix} u_{i1}(\bar{y}) \\ \vdots \\ u_{im}(\bar{y}) \end{pmatrix} \qquad \begin{pmatrix} v_{i1}(\bar{y}) \\ \vdots \\ v_{im}(\bar{y}) \end{pmatrix}$$

same degree k

same degree $d-k$

$k \leq d-k \quad (k \leq d/2)$

Upper bound on generic rank

- Note that we can break up any matrix in the form $L \times \mathbb{F}^m + \mathbb{F}^m \times L'$

$$\sum M_{\bar{e}} \bar{y}^{\bar{e}} = \boxed{M(y)} = \sum_{i=1}^{r} \underbrace{u_i(\bar{y})}_{\deg \leq \frac{d}{2}} \underbrace{v_i(\bar{y})^T}_{\deg \geq d/2}$$

$$= \sum_{i=1}^{r} \Big(\sum_{|\bar{a}|=k} \bar{y}^{\bar{a}} \cdot \underbrace{u_{i\bar{a}}}_{\uparrow\ \mathbb{F}^m} \Big) v_i(\bar{y})^T$$

$$\begin{pmatrix} xy \\ x^2 \end{pmatrix} = xy \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$n^{d/2}$$

$$\downarrow$$

$$U = \{ u_{i\bar{a}} \} \qquad \dim(\mathrm{span}(U)) \leq \# U \leq r \cdot \begin{pmatrix} \#\text{mon.} \\ \deg \\ \leq d/2 \end{pmatrix}$$

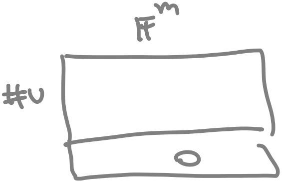
Barrier

- Linearity now bounds the rank of *any* matrix in image of map!

$$\underline{\sum M_{\bar e}\, \bar y^{\bar e}} = \sum_{i=1}^{r} \sum_{|\bar a| \le d/2} \bar y^{\bar a} \cdot u_{i\bar a}\, v_i(\bar y)^T$$

$U \subset \mathbb{F}^m$

$$\Rightarrow M_{\bar e} = \sum \underbrace{u_{i\bar a}}_{\in U} \cdot \underbrace{v_{i\bar a}^T}_{\text{anything}} \subset \mathrm{span}(U) \otimes \mathbb{F}^m$$



$\Rightarrow$ all $M_{\bar e}$ and linear combinations have low rank!

Barrier

$$f = \sum f_{\bar{e}} \bar{x}^{\bar{e}}$$

$$L(f) = \sum f_{\bar{e}} \cdot \mu_{\bar{e}}$$

$$\text{rank}\left(\sum \alpha_{\bar{e}} \mu_{\bar{e}} \right) \leq \dim(U) \leq r \cdot n^{d/2}$$

$$\Rightarrow \mu(f) \leq r \cdot n^{d/2} \quad \text{upper bd}$$

on measure for <u>any</u> polynomial f.

$$\frac{\mu(f)}{\mu(S)} \leq \frac{r \cdot n^{d/2}}{r} = n^{d/2}.$$

$$L : \mathbb{F}[\bar{x}]_d \longrightarrow \mathbb{F}^{m\times m}$$

$$\underline{\text{linear}}$$

$$L(\bar{x}^{\bar{e}}) = M_{\bar{e}}$$

$$M(\bar{y}) = L\left((y_1x_1 + \dots + y_nx_n)^d\right) =$$

$$= \underbrace{\sum_{|\bar{e}|=d} \bar{y}^{\bar{e}} \cdot \underset{\mathbb{F}}{\underset{\in}{(\quad)}} \cdot \underset{\mathbb{F}^{m\times m}}{\underset{\in}{M_{\bar{e}}}}}_{\text{degree } d}$$

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natural proofs

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- In [Razborov & Rudich 1997] they show that (under cryptographic assumptions) natural proofs *cannot* yield super-polynomial boolean circuit lower bounds!
  - would contradict existence of cryptographic pseudorandom functions.

Algebraic Natural Proofs [Forbes & Shpilka & Volk 2018, Grochow et al. 2017]

- What would be an algebraic "natural" proof?

  Property P given by an *algebraic variety* that is $\underbrace{\textit{easy to compute}}_{\text{constructive}}$: that is matrix $M : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}^{m\times m}$ such that

$$p \text{ has property P} \Leftrightarrow \underbrace{\det(M(\mathit{coeff}(p))) = 0}$$

$$f = \sum f_{\bar{e}} \bar{x}^{\bar{e}} \longmapsto \underbrace{\sum f_{\bar{e}} M_{\bar{e}}}_{M(\mathrm{coeff}(f))}$$

easy poly $\in V(\det(M(\bar{y})))$

find hard poly outside

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  1. *Useful*: for easy polynomials, $M(\mathit{coeff}(p))$ is singular

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  $m = \mathrm{poly}(N)$

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  1. *Useful*: for easy polynomials, $M(\mathit{coeff}(p))$ is singular
  2. *Constructive*: one can decide in time poly(N)-time whether p has property P.

     $\binom{n+d}{d}$

     This amounts to being able to compute $\det(M((\mathit{coeff}(p)))$, which is poly($N$)-size if $\dim(M) = \mathrm{poly}(N)$.

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  2. *Constructive*: one can decide in time poly(N)-time whether p has property P.

     This amounts to being able to compute $\det(M((\mathit{coeff}(p)))$, which is poly($N$)-size if $\dim(M) = \text{poly}(N)$.
  3. *Largeness*: Most polynomials are hard.
     This is *intrinsic* in the case of polynomials, since we know that the zero set of an algebraic variety has *measure zero*.

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$$f(\bar{x}) = \sum y_{\bar{e}} \bar{x}^{\bar{e}}$$

variables of coeff $(\mathcal{C})$

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- A polynomial $D \in \mathcal{D}$ (distinguisher) is a *algebraic natural proof* against $\mathcal{C}$ if
  1. $D \not\equiv 0$
  2. $D(coeff(f)) = 0$ for all $f \in \mathcal{C}$

$ax^2 + bxy + c$

$\mathcal{C} \subset V(D)$

$\mathcal{C} \subset \mathbb{F}[x, y]_2$ squares $(\alpha x + \beta y)^2$

$\mathcal{D} = \{ b^2 - 4ac \}$ $\qquad$ $\mathcal{C} \subset V(b^2 - 4ac)$

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Open Question (Existence of natural proofs)

Is VP a natural proof for VP?

$\mathcal{C} = VP$

$\mathcal{D} = VP$
$T(\bar{y})$

$VP(\bar{x}) \subset V(T)$
?

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Open Question (Existence of natural proofs)

Is VP a natural proof for VP?

- Question above is open even under any assumptions.
- In [KRST'20] the authors proved that if Per requires circuits of exponential size, then VP is *not* an algebraic natural proof against VNP.

When will a natural proof fail? Succinct Hitting sets

$\mathcal{C} \leftarrow$ easy polynomials

$\mathcal{D} \subset \mathbb{F}[\mathrm{coeff}(\mathcal{C})] \leftarrow$ distinguisher

$\mathcal{D}$ is NOT algebraic natural proof against $\mathcal{C}$

$\iff \forall D \in \mathcal{D} \quad \exists f \in \mathcal{C}$ s.t.

$D(\mathrm{coeff}(f)) \not\equiv 0$

$\iff \mathcal{C}$ is hitting set for $\mathcal{D}$

$\mathcal{H} = \{\mathrm{coeff}(f) : f \in \mathcal{C}\}$

Succinct Hitting Sets

- Lower bound approaches - Rank Methods
- Barriers to Rank Methods
- Algebraic Natural Proofs & Succinct PIT
- **Conclusion**
- Acknowledgements

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- Saw barriers to proving non-trivial Waring Rank
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  2. What are the connections between this line of work and *cactus rank* of varieties?
  3. More generally, can other notions of rank help us in proving lower bounds?
- Algebraic Natural Proofs
- Existence of algebraic natural proofs implies it may be harder to find succinct hitting sets (so PIT may have to be solved using more complex methods)
- Relationship between algebraic natural proofs and problems in algebraic geometry?

Acknowledgement

- Lecture based largely on:
  - [Efremenko et al. 2018]
  - [Forbes & Shpilka & Volk 2018]

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