Lecture 4: Polynomial Identity Testing

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Overview

- Word Problems and Polynomial Identity Testing
- Why is PIT so fundamental?
- PIT for restricted circuit classes
- Conclusion
- Acknowledgements

Word Problems

(1) G : multiplication table size $|G|^2$

$g_{i_1} \cdots g_{i_t} (g_{a_1} \cdots g_{a_n})^{-1} = id_G$

1. **Setting:** a group is given *succinctly* via generators and relations
2. **Input:** given a sequence of generators and operations among them forming a *word*, is this word the identity element in the group?

(2) G : generators $g_1, \ldots, g_n$

relations of G $\quad g_i g_j = g_j g_i$ (abelian)

representation 2 is more succinct than 1

$g_{i_1} g_{i_2} \cdots g_{i_t} \stackrel{?}{=} id_G$

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finitely many generators

finitely generated relations

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4. For hyperbolic groups, it is in P given Gromov's geometric techniques
5. what other word problems appear in TCS?

Polynomial Identity Testing (PIT)

word problem in algebraic complexity

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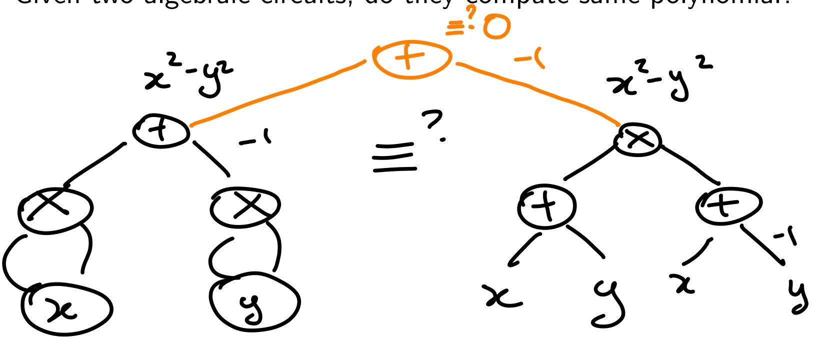

Polynomial Identity Testing (PIT)

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Polynomial Identity Testing

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Polynomial Identity Testing

4. Two ways in which input can be given:
   1. **White-box model:** circuit is given as an input, with bound on the degree of the polynomial being computed
   2. **Black-box model:** one is given a bound on the degree of the polynomial, and one has only "oracle access" via evaluation

white-box : can see the entire ckt (we can modify the ckt in ways we want)

$$\bar{x} \equiv \boxed{\Phi} \longrightarrow \Phi(\bar{x})$$

Polynomial Identity Testing (PIT)

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5. Central question in TCS
    - best parallel algorithms for finding perfect matchings
    - Primes is in P
    - used in IP = PSPACE
    - proof of PCP theorem
    - structure of algebraic proof systems

Ore-Schwartz-Zippel-deMillo-Lipton Folklore Lemma

Lemma

If $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ is a non-zero polynomial of degree $\leq d$ and $S \subset \mathbb{F}$ is a finite set, then

$$\Pr_{\underbrace{a_i \in S}}[p(a_1, \ldots, a_n) = 0] \leq \frac{d}{|S|}$$

random point $(a_1, a_2, \ldots, a_n) \in S^n$

evaluate $p(x_1, \ldots, x_n)$

p can be computed by small algebraic ckts

Gives us randomized algorithm for PIT! coRP $\subset$ BPP

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- Proof idea: in a domain $R[x]$, any polynomial $f(x)$ of degree $\leq d$ has at most d roots in R.

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- Proof idea: in a domain $R[x]$, any polynomial $f(x)$ of degree $\leq d$ has at most d roots in R. base case
- Induction on number of variables: write

$$p(x_1, \ldots, x_n) = \sum_{e=1}^{k} \underline{p_e(x_1, \ldots, x_{n-1})} x_n^e \qquad p_k \neq 0$$

$\deg_n(p) = k$

Ore-Schwartz-Zippel-deMillo-Lipton Folklore Lemma

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- Proof idea: in a domain $R[x]$, any polynomial $f(x)$ of degree $\leq d$ has at most d roots in R.
- Induction on number of variables: write

$\deg(p) \leq d$

$\deg(p_k \cdot x_n^k) \leq d$

$$p(x_1, \ldots, x_n) = \sum_{e=1}^{k} p_e(x_1, \ldots, x_{n-1}) x_n^e \qquad p_k \neq 0$$

$\deg(p_k) \leq d-k$

- By induction hypothesis $\Pr_{a_i \in S}[p_k(a_1, \ldots, a_{n-1}) = 0] \leq \dfrac{d-k}{|S|}$
- If $p_k(a_1, \ldots, a_{n-1}) \neq 0$ then $\leq k$ values of x_n will make $\underbrace{p(a_1, \ldots, a_{n-1}, x_n)}_{\deg k}$ zero, as it has degree k.

Black-Box Setting: Hitting Sets & Generators

$P(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$

- In black-box setting, given a circuit class $\mathcal{C}$, all we can do is to come up with a set $\mathcal{H} \subset \mathbb{F}^n$ (*hitting set*) such that

$$\Phi \in \mathcal{C}, \quad \Phi \not\equiv 0 \quad \Rightarrow \quad \exists \alpha \in \mathcal{H} \ \text{ s.t. } \ \Phi(\alpha) \neq 0$$

$\mathcal{H}$ hitting set ckt class $\mathcal{C}$

if $\Phi \in \mathcal{C}$ $\Phi \not\equiv 0$

then there is pt $\alpha \in \mathcal{H}$

s.t. $\Phi(\alpha) \neq 0$

(witnesses non-zeroness)

$\alpha_1 \to \Phi(\alpha_1)$

"

0

$\Phi(\alpha_2) = 0$

$\Phi(\alpha_3) = 0$

"non-adaptive setting"

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- A polynomial map $\mathcal{G} = (\overbrace{g_1, \ldots, g_n}^{\text{polynomials}}) : \mathbb{F}^t \to \mathbb{F}^n$ is a *hitting set generator* for a circuit class $\mathcal{C}$ if

$$\Phi(x_1, \ldots, x_n) \in \mathcal{C}, \quad \Phi \not\equiv 0 \quad \Rightarrow \quad \underbrace{[\Phi \circ (g_1, \ldots, g_n)]}_{\text{polynomial}}(y_1, \ldots, y_t) \not\equiv 0$$

non zero

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$$\Phi(x_1, \ldots, x_n) \in \mathcal{C}, \quad \Phi \not\equiv 0 \quad \Rightarrow \quad \underbrace{[\Phi \circ (g_1, \ldots, g_n)]}_{D \leftarrow \text{poly}}(y_1, \ldots, y_t) \not\equiv 0$$

- Hitting set generator decreases number of variables, and we can use brute-force to find non-zero

$|S| = D+1$ evaluate on S^t

$(D+1)^t \sim (D+1)^{\log n} \sim n^{\log n}$

$t = \log n$ quasi-P

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- Hitting set generator decreases number of variables, and we can use brute-force to find non-zero
- In algebraic complexity, hitting set generators are also pseudorandom generators (decreased the number of "random seeds" needed)

$t \ll n$ means reducing randomness needed in Schwartz-Zippel

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- PIT is an outstanding open question in derandomization (understand whether randomness is needed in design of efficient algorithms)

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- Hardness-Randomness tradeoff:

Theorem ([Kabanets & Impagliazzo 2004])

The following three assumptions cannot be simultaneously true:

1. $NEXP \subseteq P_{/poly}$
2. *Permanent is computable by polynomial size arithmetic circuits over* $\mathbb{Z}$
3. $PIT \in SUBEXP$

if have efficient PIT algorithm then we proved a lower bd that is way beyond current reach!

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- Today we will show that (a strong version of) $\neg 2 \Rightarrow 3$

$$\text{Exponential lower bound on Permanent} \Rightarrow PIT \in \text{quasi-P}$$

hardness $\Rightarrow$ derandomization

"replace randomness by hard function"

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Lower Bounds imply Derandomization

NW'97 hardness vs randomness
in Boolean comp.

- Use Nisan-Wigderson designs:
  - $n \leq 2^m$ integers
  - There exist $S_1, \ldots, S_n \subset [m^2]$ such that
  - $|S_i| = m$, for all $1 \leq i \leq k$
  - $i \neq j \Rightarrow |S_i \cap S_j| \leq \log(n)$

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    - $i \neq j \Rightarrow |S_i \cap S_j| \leq \log(n)$
- relaxed notion of combinatorial designs

S_i large enough but have small pairwise intersection

(q, k, t)-design

$[m] \supset S_1, \ldots, S_n$

$|S_i| = q$

$i \neq j \quad |S_i \cap S_j| = k \quad (\leq k)$

drop $\{$ $e \in [m]$ appears in t subsets $\}$

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  2. Then, $\mathbb{F}_p^2 \sim [m^2]$

$(a, b) \in \mathbb{F}_p^2$

$\mathbb{F}_p$ finite field

p elements

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    3. Let $q_1, \dots, q_n \in \mathbb{F}_p^{\log(n)}$ be polynomials of degree $< \log(n)$.

$p^{\log n} > n$

$q_i(x) = q_{i0} + q_{i1}x + \dots + q_{i(\log n - 1)} x^{\log n - 1}$

$(q_{i0}, q_{i1}, \dots, q_{i(\log n - 1)})$

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  4. $S_i = \{(a, q_i(a)) \mid a \in \mathbb{F}_p\}$ $\Rightarrow |S_i| = |\mathbb{F}_p| = p$

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  5. $(a, y) \in \underbrace{S_i \cap S_j}_{} \Leftrightarrow \underbrace{q_i(a)}_{} = \underbrace{q_j(a)}_{} = y$

$(a, y) = (a, q_i(a)) = (a, q_j(a))$

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  5. $(a, y) \in S_i \cap S_j \Leftrightarrow q_i(a) = q_j(a) = y$
  6. $|S_i \cap S_j| \leq \deg(q_i) < \log(n)$

$\deg(q_i - q_j) < \log n$

at most $\log n$ roots

$(q_i - q_j)(a) = 0$

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- Take NW-design with $m = \log^4 n$ $(n < 2^m)$
  - $S_1, \ldots, S_n \subset [m^2]$
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- Hitting set generator: $\mathcal{G} = (g_1, \ldots, g_n) : \mathbb{F}^{m^2} \to \mathbb{F}^n$

poly log(n) ↓

quasiP alg. PIT

$\log^2 n = \sqrt{m}$

$$g_i(y_{S_i}) = \boxed{\mathrm{Per}_{\log^2 n}(y_{S_i})}$$

depends only on the variables in S_i

$g_i(y_{S_i})$ poly in $\log^4 n$ variables
$\log^2 n$ deg.

Can have quasi-P many monomials
(write it in sparse representation)

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- Hitting set generator: $\mathcal{G} = (g_1, \dots, g_n) : \mathbb{F}^{m^2} \to \mathbb{F}^n$

$$g_i(y_{S_i}) = \mathrm{Per}_{\log^2 n}(y_{S_i}) \qquad S(\Phi) = n^c$$

$\Rightarrow$

- For any $\Phi \in \mathrm{VP}$ we have $\Phi \equiv 0 \Leftrightarrow \Phi \circ \mathcal{G} \equiv 0$

($\Leftarrow$)

1. Suppose $\Phi \not\equiv 0$ but $\Phi \circ \mathcal{G} \equiv 0$ — Hybrid argument
2. There is index $k \in [n]$ such that $\Phi(g_1, \dots, g_k, x_{k+1}, \dots, x_n) \not\equiv 0$ but $\Phi(g_1, \dots, g_k, g_{k+1}, x_{k+2}, \dots, x_n) \equiv 0$

$x_{k+1} \leftarrow g_{k+1}$

$\boxed{x_{k+1} - g_{k+1}}$ root of $\Phi(g_1, \dots, g_k, x_{k+1}, \dots, x_n)$

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  1. Suppose $\Phi \not\equiv 0$ but $\Phi \circ \mathcal{G} \equiv 0$
  2. There is index $k \in [n]$ such that $\Phi(g_1, \ldots, g_k, x_{k+1}, \ldots, x_n) \not\equiv 0$ but $\Phi(g_1, \ldots, g_k, g_{k+1}, x_{k+2}, \ldots, x_n) \equiv 0$
  3. $x_{k+1} - g_{k+1}$ divides $\Phi(g_1, \ldots, g_k, \cancel{g_{k+1}}, x_{k+2}, \ldots, x_n)$

x_{k+1}

because g_{k+1} root .

x_{k+1} , $y_{S_{k+1}}$ important variables

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  3. $x_{k+1} - g_{k+1}$ divides $\Phi(g_1, \dots, g_k, g_{k+1}, x_{k+2}, \dots, x_n)$
  4. Set variables $\underbrace{x_{k+2}, \dots, x_n}_{\text{other x vars}}$, and $\underbrace{y_j \in [m^2] \setminus S_{k+1}}_{\text{other y variables}}$ to random values

$x_{k+1} - g_{k+1}$ unchanged Φ remain nonzero (still divisible by $x_{k+1} - g_{k+1}$)

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  1. Suppose $\Phi \not\equiv 0$ but $\Phi \circ \mathcal{G} \equiv 0$
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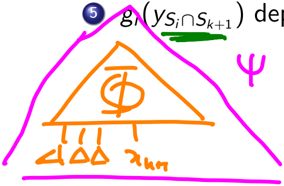
  3. $x_{k+1} - g_{k+1}$ divides $\Phi(\underline{g_1}, \ldots, \underline{g_k}, \overset{x_{k+1}}{\cancel{g_{k+1}}}, x_{k+2}, \ldots, x_n)$
  4. Set variables $x_{k+2}, \ldots, x_n$, and $y_j \in [m^2] \setminus S_{k+1}$ to random values
  5. $g_i(\underline{y_{S_i \cap S_{k+1}}})$ depends only on $\log n$ variables, so poly-size circuit!



$|S_i \cap S_{k+1}|$

$\log^2 n$ deg.

$s(\Psi) = n^c$

Lower Bounds imply Derandomization

- Assume Per_n cannot be computed by circuits of size $\leq 2^{cn}$
- Take NW-design with $m = \log^4 n$
  - $S_1, \ldots, S_n \subset [m^2]$
  - $|S_i| = m$ and $|S_i \cap S_j| \leq \log n$
- Hitting set generator: $\mathcal{G} = (g_1, \ldots, g_n) : \mathbb{F}^{m^2} \to \mathbb{F}^n$

$$g_i(y_{S_i}) = \mathrm{Per}_{\log^2 n}(y_{S_i})$$

any factor of Φ has ckt size $n^{\tilde{c}}$

- For any $\Phi \in \mathrm{VP}$ we have $\Phi \equiv 0 \Leftrightarrow \Phi \circ \mathcal{G} \equiv 0$
  1. Suppose $\Phi \not\equiv 0$ but $\Phi \circ \mathcal{G} \equiv 0$
  2. There is index $k \in [n]$ such that $\Phi(g_1, \ldots, g_k, x_{k+1}, \ldots, x_n) \not\equiv 0$ but $\Phi(g_1, \ldots, g_k, g_{k+1}, x_{k+2}, \ldots, x_n) \equiv 0$
  3. $x_{k+1} - g_{k+1}$ divides $\Phi(g_1, \ldots, g_k, g_{k+1}, x_{k+2}, \ldots, x_n)$
  4. Set variables $x_{k+2}, \ldots, x_n$, and $y_j \in [m^2] \setminus S_{k+1}$ to random values
  5. $g_i(y_{S_i \cap S_{k+1}})$ depends only on $\log n$ variables, so poly-size circuit!
  6. By Kaltofen, VP is closed under taking factors
  7. Implies $x_{k+1} - g_{k+1}$ has poly size circuit!

Φ poly deg poly size ckts $\Rightarrow$ any factor of Φ also has poly-size ckts

Lower Bounds imply Derandomization

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  6. By Kaltofen, VP is closed under taking factors
  7. Implies $\underline{y - g_{k+1}}$ has poly size circuit! $S(g_{k+1}) \leq n^c = 2^{c \log n}$
  8. Contradicts fact that $\mathrm{Per}_{\log^2 n}$ cannot be computed by $2^{c \log^{2} n} = n^{c \log n}$ size

Lower Bounds imply Derandomization

$x_{k+1} - g_{k+1} \mid \Phi(g_1, \ldots, g_k, x_{k+1}, \ldots, x_n)$

set $x_{k+2}, \ldots, x_n$ $\left.\begin{matrix} \\ \\ y_i \in [m^2] \setminus S_{k+1} \end{matrix}\right\}$ random field elements

$g_i(y_{S_i}) \xrightarrow[\text{restriction}]{\text{after}} g_i(\underbrace{y_{S_i \cap S_{k+1}}}_{\substack{\text{alive vars} \\ \leq \log n}})$

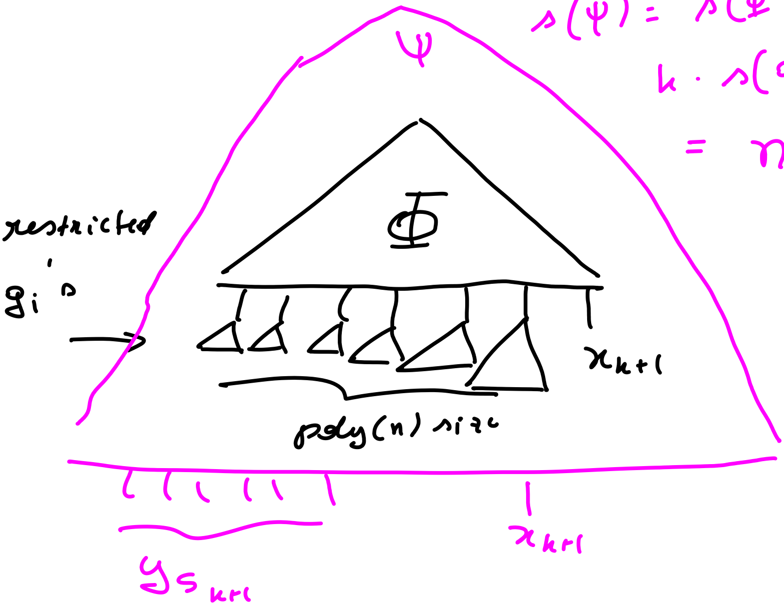
g_i poly in $\log n$ vars
~~deg $\log^2 n$~~
multilinear

only monomials that are left $\boxed{\prod_{\ell \in T} y_\ell}$ $T \subset [S_i \cap S_{k+1}]$

have $\leq 2^{c \log n} = n^c$ monomials

Lower Bounds imply Derandomization

$$s(\Psi) = s(\Phi) + k \cdot s(g_i) = n^c$$



Fast Parallel Algorithms for Matching

Fast Parallel Algorithms for Matching

Sparse Polynomials - Klivans-Spielman

- **Input:** oracle (black-box) access to a polynomial $p(x_1, \ldots, x_n)$ with $\leq s$ monomials and degree d $(n, s, d$ given to you)
- **Output:** is $p(x_1, \ldots, x_n) \equiv 0$?

poly(n, s, d)

$\Sigma \Pi$

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- First idea: Kronecker substitution

preserves all monomials

$$P\left(y^{d+1}, y^{(d+1)^2}, \ldots, y^{(d+1)^n}\right) \not\equiv 0$$

$$\Updownarrow$$

$$P(x_1, \ldots, x_n) \not\equiv 0$$

"base $d+1$"

$e_i \leq d$

$x_1^{e_1} \cdots x_n^{e_n}$

$$P(\bar{x}) = \sum p_{\bar{e}} \cdot \bar{x}^{\bar{e}}$$

$$x_i \mapsto y^{(d+1)^i}$$

base $d+1$

$$(e_1, \ldots, e_n) \longrightarrow \sum_{i=1}^{n} e_i (d+1)^i$$

$(a_1, \ldots, a_n)$

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- Problem is that the degree is really high. How to fix it?
- Let $p \in \mathbb{Z}$ be a prime. Make substitution:

$$x_i \to y^{\underline{(d+1)^i \mod p}}$$

deg var $\leq p$

- Now degrees are under control. But how to preserve non-zeroness?

$\exists$ monomials can be mapped to the same univariate monomial y

Sparse Polynomials - Klivans-Spielman

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- Problem is that the degree is really high. How to fix it?
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$$x_i \to y^{(d+1)^i} \mod p$$

- Now degrees are under control. But how to preserve non-zeroness?
- Chinese Remaindering Theorem!
  1. If two monomials $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are distinct and degree $\leq d$, then

     $$a_1 + a_2(d+1) + \cdots + a_n(d+1)^n \neq b_1 + b_2(d+1) + \cdots + b_n(d+1)^n$$

  2. Thus if we take $p_1, \ldots, p_{nd}$ primes, one of the differences $\mod p_i$ will be non-zero

poly many primes

Pick enough primes and do union bound

Sparse Polynomials - Klivans-Spielman

$p_1, \ldots, p_m$ $\quad m = \text{poly}(n, s, d)$

there is one which preserves all s monomials

$P\left(y^{d+1 \bmod p_i}, \ldots, y^{(d+1)^n \bmod p_i}\right) \not\equiv 0$

and has $\leq p_i d$ roots

test univariate poly over $\{0, \ldots, p_i d\}$

Sparse Polynomials - Klivans-Spielman

Conclusion

- Today we learned about word problems and their importance
- Polynomial Identity Testing (PIT)
- Hardness versus randomness
- Application of PIT in TCS (parallel algorithms for matching)
- deterministic PIT algorithm for sparse polynomials

Acknowledgement

- Lecture based largely on:
  - Survey [Shpilka & Yehudayoff 2010, Chapter 4]

References I

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