Lecture 3: Lower Bounds in Algebraic Complexity

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Overview

- “Natural” lower bound strategies
- Lower Bounds for Homogeneous Depth-3 Circuits
- Shifted Partial Derivatives and Depth-4 Circuits
- Conclusion
- Acknowledgements

How to prove lower bounds?

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   - μ is sub-additive and sub-multiplicative

$$\mu(f + g) \leq \mu(f) + \mu(g)$$

not so important

$$\mu(f \cdot g) \leq \mu(f) \cdot \mu(g)$$

$$\boxed{\mu(f) + \mu(g)}$$

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$$\mu(f + g) \leq \mu(f) + \mu(g)$$

$$\mu(f \cdot g) \leq \mu(f) + \mu(g) + 1$$

   - μ is "easy" to compute or estimate } useful
3. For instance $\boxed{S(f)}$ is a valid complexity measure, but we don't know how to estimate it, let alone compute it

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3. For instance $S(f)$ is a valid complexity measure, but we don't know how to estimate it, let alone compute it
4. *Open problem:* what is the complexity of computing $S(f)$, if I give f in dense representation? (algebraic minimum circuit size problem)

f deg d n vars $\binom{n+d}{d}$ poly$\left(\binom{n+d}{d}\right)$ time

How have we usually proved lower bounds?

$s_n : \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{N}$

1. Define class of simple polynomials $\mathcal{S}$
2. *Normal form:* every circuit from circuit class $\mathcal{C}$ can be expressed as small sum of simple polynomials in $\mathcal{S}$

problem: given circuit class $\mathcal{C}$, parameter h_n

find explicit polynomial $\{P_n\}_{n \geq 1}$ s.t. P requires

size h_n in class $\mathcal{C}$

$$\Phi \in \mathcal{C} \qquad \Phi = P \Rightarrow s(\Phi) \geq h$$

$$\{\mathrm{Det}_n\}_{n \geq 1} \qquad \{\mathrm{Per}_n\}_{n \geq 1} \qquad \{E_{n,d}\}_{n,d \geq 1}$$

$d = d(n)$

$\mathcal{C}$ = poly-sized formulas } we choose

$\mathcal{G}$ = $f \cdot g$ where

$$\frac{1}{3}\deg(fg) \le \deg(f), \deg(g) \le \frac{2}{3}\deg(fg)$$

} we choose

$\Phi \in \mathcal{C}$ $\quad s(\Phi) = s$

$$\Phi = \sum_{i=1}^{s} f_i g_i$$

any formula of size s can be written as sum of $\le s$ "simple" polynomials

Hyafil '70s

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2. *Normal form:* every circuit from circuit class $\mathcal{C}$ can be expressed as <u>small sum of simple polynomials</u> in $\mathcal{S}$
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$f \in S \Rightarrow \mu(f)$ small $(\leq U)$

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4. *Hard polynomial:* find polynomial p such that $\mu(p)$ is large

- If $\mu(f) \leq U$ for all $f \in \mathcal{S}$
- By sub-additivity $\mu(q) \leq s \cdot U$ for any $q \in \mathcal{C}$ which can be written as

$$\boxed{q = \sum_{i=1}^{s} f_i,} \quad f_i \in \mathcal{S}$$

$$\Rightarrow \mu(q) \leq \sum_{i=1}^{s} \mu(f_i) \leq s \cdot U$$

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- By sub-additivity $\mu(q) \leq s \cdot U$ for any $q \in \mathcal{C}$ which can be written as
$$\overset{p}{\cancel{q}} = \sum_{i=1}^{s} f_i, \quad f_i \in \mathcal{S}$$

$L \leq \mu(p) \leq s \cdot U$

$s \geq L/U$

- $\boxed{\mu(p) \geq L}$ and p can be computed by size s in $\mathcal{C} \Rightarrow s \cdot U \geq L$

Lower bounds

Why study constant depth circuits?

$\Sigma\Pi\Sigma$ homogeneous

$\Sigma\Lambda^d(\underbrace{\Sigma\Pi^2}_{\text{quadratic polynomials}})$

$\Sigma\Pi^{\sqrt{d}}\Sigma\Pi^{\sqrt{d}}$

- AV'08, GKKS'12,13, Koi'10s, Tav'13

depth reduction

P computed by circuits of size s

$\Rightarrow$ P computed by depth-4 homogeneous ckts (depth-3 non-homogeneous) of size $n^{\sqrt{s}}$

if we prove strong enough lower bds for depth-3/4 ckts $\Rightarrow$ general ckt lbs.

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Homogeneous depth-3 circuits

- Simple polynomials: products of linear forms

$$\boxed{\prod_{i=1}^{d}(a_{i1}x_1 + \cdots + a_{in}x_n)}$$

$$\Sigma\,\Pi^{[d]}\,\Sigma$$

$$P = \sum_{i=1}^{s} \underbrace{\prod_{j=1}^{d} l_{ij}(\bar{x})}$$

Homogeneous depth-3 circuits

- Simple polynomials: products of linear forms

$$\prod_{i=1}^{d}(a_{i1}x_1 + \cdots + a_{in}x_n)$$

- Circuit class $\mathcal{C}$

$$\mathcal{C}(s, n, d) := \left\{ f \in \mathbb{F}[x_1, \ldots, x_n]_d \mid f = \sum_{i=1}^{s}\prod_{j=1}^{d}\ell_{ij}(x_1, \ldots, x_n) \right\}$$

homogeneous
poly of degree d

Homogeneous depth-3 circuits

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- Complexity measure: dimension of space of all partial derivatives

$$\mu(f) = \dim(\partial^* f)$$

Homogeneous depth-3 circuits

$\partial_j x_i = \delta_{ij}$

x_i

- Simple polynomials: products of linear forms

$$\prod_{i=1}^{d}(a_{i1}x_1 + \cdots + a_{in}x_n)$$

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- Complexity measure: dimension of space of all partial derivatives

$$\mu(f) = \dim(\partial^* f)$$

- Examples:

$$\mu(x_i) = 2 \qquad\qquad \mu(x_1 \cdot x_2 \cdots x_n) = 2^n$$

$\dim(\mathbb{F}\text{-span}\{1, x_i\}) = 2$ $\quad\dim\langle x_S\rangle_{S\subseteq[n]} = 2^n$

Property of partial derivatives

$\partial f \qquad \partial(\alpha f) = \alpha \cdot \partial f$

Given polynomials $f, f_1, \ldots, f_k \in \mathbb{F}[x_1, \ldots, x_n]$, $\alpha \in \mathbb{F}^*$ we have:

- $\dim(\partial^*(\alpha f)) = \dim(\partial^* f)$ ✓
- $$\dim\left(\partial^*\left(\sum_{i=1}^{k} f_i\right)\right) \leq \sum_{i=1}^{k} \dim\left(\partial^* f_i\right)$$ subadditive
- $$\dim\left(\partial^*\left(\prod_{i=1}^{k} f_i\right)\right) \leq \prod_{i=1}^{k} \dim\left(\partial^* f_i\right)$$

$\underbrace{x_i^2}_{1,\ x_i,\ x_i^2}$

have dependencies

$\dim(U+V) \leq \dim(U) + \dim(V)$

3

$\mu(x_i) \cdot \mu(x_i) = 2 \cdot 2$

Lower Bound

Theorem ([Nisan & Wigderson 1997])

Any depth-3 homogeneous circuit computing the elementary symmetric polynomial $E_{n,2d}$ must be of size

$$\left(\frac{n}{4d}\right)^d$$

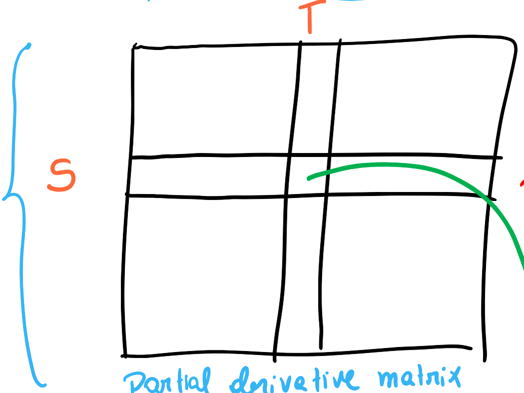
$$\mu\left(\prod_{i=1}^{2d} \ell_i\right) \leq \prod_{i=1}^{2d} \mu(\ell_i) = \prod_{i=1}^{2d} 2 = 2^{2d} = 4^d$$

- upper bound on measure for simple polynomials.

Lower Bounding $\mu(E_{n,2d})$

$$\dim\Big(\underbrace{\partial^{*}(E_{n,2d})}\Big) \geq \dim\Big(\partial^{=d}(E_{n,2d})\Big)$$

coeff of the polynomial



Partial derivative matrix

partial derivative

$S, T \subset \binom{[n]}{d}$

$\partial_S\, \alpha_{S,T} \cdot x_S x_T$

$= \alpha_{S,T} \cdot x_T$

coeff of $x_S \cdot x_T$ in $E_{n,2d}$

$x_S = \prod_{i \in S} x_i$

Lower Bounding $\mu(E_{n,2d})$

$2d = 2$

$$E_{3,2} = x_1x_2 + x_1x_3 + x_2x_3$$

$$\begin{array}{c|ccccccc}
 & \begin{matrix}x_1x_2\\1,2\end{matrix} & \begin{matrix}x_1x_3\\13\end{matrix} & \begin{matrix}x_2x_3\\23\end{matrix} & \begin{matrix}x_1\\1\end{matrix} & \begin{matrix}x_2\\2\end{matrix} & \begin{matrix}x_3\\3\end{matrix} & \begin{matrix}1\\\emptyset\end{matrix} \\
\hline
\emptyset & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
\{1\} & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\{2\} & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\{3\} & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\{1,2\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{1,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\{2,3\} & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}$$

$1\cdot x_1x_2 + 1x_1x_3 + 1\cdot x_2x_3$

$x_2 + x_3 = \partial_1 E_{3,2}$

$x_1 + x_3$

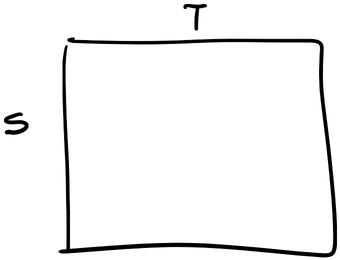
$\partial_3 E_{3,2} = x_1 + x_2$

M

$\mathrm{rank}(M) = \dim(\partial^* f) \geq \mathrm{rank}(\text{submatrix})$

Lower Bounding $\mu(E_{n,2d})$

Problem: rank submatrix



$|S| = |T| = d$

$S, T \in \binom{[n]}{d}$

Comm. complexity [Kushilevitz-Nisan]
Pages 23, 24

rank of this submatrix is full
(induction)

if $p \in \Sigma^{[s]} \Pi^{[2d]} \Sigma$ then

$$\mu(p) \le 4^d \cdot s$$

upper bd on measure

$$\mu(E_{n,2d}) \geq \binom{n}{d} \geq \left(\frac{n}{d}\right)^d$$

if $E_{n,2d} \in \Sigma^{[s]} \Pi^{[2d]} \Sigma$

then

$$\left(\frac{n}{d}\right)^d \le s \cdot 4^d \Rightarrow s \geq \left(\frac{n}{4d}\right)^d \quad \square$$

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Our Circuit Class

- Simple polynomials: powers of quadratics Q^d, where $Q \in \mathbb{F}[x_1, \dots, x_n]_2$

$Q = x_1x_2 + x_1x_3 + x_2x_3$

Q^d simple polynomials

Our Circuit Class

- Simple polynomials: powers of quadratics Q^d, where $Q \in \mathbb{F}[x_1, \ldots, x_n]_2$
- Sums of powers of quadratic polynomials:

$$\mathcal{C}(n, d, s) = \left\{ f \in \mathbb{F}[x_1, \ldots, x_n]_{2d} \mid f = \sum_{i=1}^{s} Q_i(x_1, \ldots, x_n)^d \right\}$$

where $Q_i \in \mathbb{F}[x_1, \ldots, x_n]_2$

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- Complexity measure?
  - dimension of partial derivatives won't work, as space of partial derivatives of Q^d could be as large as can be expected
  - *Observation:* k^{th} order partial derivative of Q^d is of the form $Q^{d-k}p$, where $p \in \mathbb{F}[x_1, \ldots, x_n]_k$.
  - small order partial derivatives share *large common factors*

Complexity Measure [Kayal 2012]

- Let $\partial^k f$ be the set of all k^{th} order partial derivatives of f
- $\mathbf{x}^{\leq \ell}$ be the set of all monomials of degree $\leq \ell$

$x_1 \quad x_1^2 \quad x_1^3 \ldots \quad x_1^{\ell}$

$x_1 x_2 \quad , \quad x_1^{\ell-1} x_2$

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- $\mathbf{x}^{\leq \ell}$ be the set of all monomials of degree $\leq \ell$
- The *shifted partials* measure of f, denoted

$$\mu_{k,\ell}(f) = \dim(\mathrm{span}(\underbrace{\mathbf{x}^{\leq \ell}} \cdot \underbrace{\partial^k f}))$$

$$Q = (x_1x_2 + x_2x_3 + x_1x_3)^{d}$$

$$\mathrm{span}\left\{ \begin{array}{l} x^m \cdot (x_2+x_3)Q^{d-1} \cdot d \\ x^m \cdot (x_1+x_3)Q^{d-1} \cdot d \\ x^m \quad (x_1+x_2)Q^{d-1} \cdot d \end{array} \right\}$$

$$m = (m_1, \ldots, m_n)$$

$$\sum m_i \leq \ell$$

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Lemma (Simple polynomials have small measure)

If $f = Q^d$ where Q is a quadratic, then

$$\mu(f) \leq \binom{n+k+\ell}{n}$$

which is the number of monomials of degree $\leq k + \ell$ in n variables.

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- This measure has an algebro-geometric meaning (see Affine Hilbert function of an ideal)

Proof of Lemma

$|\bar{u}| \le \ell$

$Q \quad , \quad |S| = k$

$$\boxed{\partial_S Q^d} = \alpha_k \cdot Q^{d-k} \cdot \underbrace{\bar{X}^{\bar{m}}} \cdot \bar{x}^{\bar{u}} = Q^{d-k} \cdot \bar{x}^{\bar{m}+\bar{u}}$$

$|\bar{m}| = k = \sum_{i=1}^{n} m_i$

$\underbrace{|\bar{m}+\bar{u}|} \le k+\ell$

$= |\bar{m}| + |\bar{u}|$

$$\text{span}\left\{ \boxed{Q^{d-k} \cdot \bar{X}^{\bar{e}}} \right\} \supset \underline{\bar{X}^{\le \ell} \partial^k (Q^d)}$$

$|\bar{e}| = k+\ell$

$$\Rightarrow \dim\left(\bar{X}^{\le \ell} \partial^k (Q^d)\right) \le \#\,\text{monomials of } x_1, \ldots, x_n \text{ of degree} \le k+\ell = \binom{n+k+\ell}{n}$$

Lower Bound

Theorem ([Kayal 2012])

The monomial $x_1 x_2 \cdots x_n$ has complexity $2^{\Omega(n)}$ in the model of sums of powers of quadratics.

hard polynomial : $x_1 x_2 \cdots x_n$

Lower Bound

Theorem ([Kayal 2012])

The monomial $x_1 x_2 \cdots x_n$ has complexity $2^{\Omega(n)}$ in the model of sums of powers of quadratics.

- Lower bound $\mu_{k\ell}(x_1 \cdots x_n) \geq \underbrace{\binom{n}{k}}_{\text{\# sets of size } k} \cdot \underbrace{\binom{n-k+\ell}{\ell}}_{\text{\# monomials in } [n]\setminus S \text{ of deg.} \leq \ell}$

$S \subset [n]$ $\quad |S| = k$ $\quad$ (to take derivative)

$$\partial_S (x_1 \cdots x_n) = \underbrace{x_{[n]\setminus S} \cdot \left(\begin{array}{c}\text{monomials over}\\ \text{variables in } [n]\setminus S\\ \text{of degree} \leq \ell\end{array}\right)}_{\text{don't change support}}$$

for each $\underline{S}$ we get $\geq \binom{n-k+\ell}{\ell}$

Lower Bound

Theorem ([Kayal 2012])

The monomial $x_1 x_2 \cdots x_n$ has complexity $2^{\Omega(n)}$ in the model of sums of powers of quadratics.

- Lower bound $\mu_{k\ell}(x_1 \cdots x_n) \geq \binom{n}{k} \cdot \binom{n-k+\ell}{\ell}$
- Parameters: $\ell = 2n$, $k = \epsilon \cdot n/2$

$$x_1 \cdots x_n = \sum_{i=1}^{s} Q_i^d \qquad d = n/2$$

$$\underbrace{s \cdot \binom{n+k+\ell}{n}}_{\text{upper bd on simples}} \geq \underbrace{\binom{n}{k}\binom{n-k+\ell}{\ell}}_{\text{lower bd on hard polynomial}}$$

Conclusion

- Today we learned that constant depth circuits are essentially as general as general algebraic circuits
- Natural approaches to prove lower bounds on circuit classes
- Use of partial derivatives as a complexity measure
- Shifted partial derivatives

Acknowledgement

- Lecture based largely on:
  - Survey [Saptharishi, Chapters 7 & 13]
  - Survey [Chen, Kayal & Wigderson 2010]
  - Paper [Kayal 2012]

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