123

CS860 L24

Primary and Secondary Invariants

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Agenda

- (Re-)introducing the setting
- Hilbert series and Molien's formula
- Counting result and finding invariants

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Setting

- $\mathbb{C}[\bar{x}]$: $\mathbb{C}[x_1, \dots, x_m]$
- G : <u>finite</u> group acting on $\mathbb{C}[\bar{x}]$
- (Think of G as consisting of $\cancel{n \times n}$ $m \times m$ matrices)

$$g \in G \quad \text{maps} \quad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \quad \text{to} \quad g\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

- $\mathbb{C}[\bar{x}]^G$: polynomials invariant under G-action.

  e.g. $G = \{I_m\}$, $\mathbb{C}[\bar{x}]^G = \mathbb{C}[\bar{x}]$

  $G = S_m$, $\mathbb{C}[\bar{x}]^G = \{\text{symmetric polynomials}\}$

  $(= \mathbb{C}[\sigma_1, \dots, \sigma_m])$

About $\mathbb{C}[\bar{x}]^G$

- It is a $\mathbb{C}$-algebra.
- It is graded as follows:

$$\mathbb{C}[\bar{x}]^G = \mathbb{C}_1[\bar{x}]^G \oplus \mathbb{C}_2[\bar{x}]^G \oplus \dots \oplus \underbrace{\mathbb{C}_d[\bar{x}]^G}_{\text{degree-}d\text{ homogeneous part}} \oplus \dots$$

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Let A be a graded $\mathbb{C}$-algebra.

- Its Krull dimension is the maximal # of algebraically independent elements
- If A has Krull dimension n, $\{\theta_1, \dots, \theta_n\}$ is said to be a h.s.o.p. if A is f.g. as a $\mathbb{C}[\theta_1, \dots, \theta_n]$-module (and θ_i's are homogeneous).

[Example in class: $\mathbb{C}[x,y,z] = 1 \cdot \mathbb{C}[x,y^2,z^2] + y \cdot \mathbb{C}[x,y^2,z^2] + z \cdot \mathbb{C}[x,y^2,z^2] + yz \cdot \mathbb{C}[x,y^2,z^2]$.

$\Rightarrow \{x, y^2, z^2\}$ is a h.s.o.p. of $\mathbb{C}[x,y,z]$.

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- It has Krull dimension = # of variables
- There exists h.s.o.p. (by Noether Normalization Lemma)

Primary and Secondary Invariants of $\mathbb{C}[\bar{x}]^G$.

- Recall that a graded $\mathbb{C}$-algebra A is Cohen-Macaulay if it admits a

広中 Hironaka decomposition:

$$A = \bigoplus_{i=1}^{t} \mathbb{C}[\theta_1, \dots, \theta_n] \cdot \eta_i .$$

- When G is finite, $\mathbb{C}[\bar{x}]^G$ is Cohen-Macaulay (L13)
- For every h.s.o.p. $\{\theta_1, \dots, \theta_n\}$, there corresponds $\eta_1, \dots, \eta_t$ s.t.

$$\mathbb{C}[\bar{x}]^G = \bigoplus_{i=1}^{t} \mathbb{C}[\underbrace{\theta_1, \dots, \theta_n}_{\text{primary invariants}}] \cdot \eta_i$$

η_i: secondary invariants

GOAL OF TODAY:

Prove that $t = \dfrac{\prod_{i=1}^{n} \deg(\theta_i)}{|G|}$. (and more!)

★ This will give us useful information in understanding the invariants.

Examples

① $G = S_n$, $\mathbb{C}[\bar{x}]^G$.

$$\mathbb{C}[\bar{x}]^G = \mathbb{C}[\sigma_1, \dots, \sigma_n] \cdot 1$$

$$1 = t = \frac{\prod_{i=1}^{n} \deg(\sigma_i)}{|G|} = \frac{n!}{n!}$$

② $G = \{I_n\}$, $\mathbb{C}[\bar{x}]^G$.

$$\theta_i = x_i^{\alpha_i}, \quad i = 1, \dots, n.$$

$$t = \frac{\prod_{i=1}^{n} \deg(\theta_i)}{|G|} = \prod_{i=1}^{n} \alpha_i.$$ ← Look for that many secondary invariants.

$\{\bar{x}^{\beta} : 0 \leq \beta_i < \alpha_i\}$ will be our set of secondary invariants.

Outline of proof.

① Introduce the Hilbert series, defined for graded $\mathbb{C}$-algebras A

② Using Hironaka decomposition, derive an alternative form of the Hilbert series.

(This carries information about the invariants $\{\theta_i\}$, $\{\eta_j\}$)

③ Using the fact that $A = \mathbb{C}[\vec{x}]^G$, derive yet another form of the Hilbert series (Molien series)

(This carries information about the group G.)

④ Equate the two. Profit! :)

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Hilbert series

Def. Given A: f.g. graded $\mathbb{C}$-algebra, with decomposition

$$A = \bigoplus_{d \geq 0} A_d \quad , \quad A_0 = \mathbb{C} .$$

The Hilbert series of A (denoted by Φ_A), is defined as

$$\Phi_A(z) := \sum_{d=0}^{\infty} \underbrace{\dim_{\mathbb{C}}(A_d)}_{\text{dimension as } \mathbb{C}\text{-vector space}} z^d .$$

$\boxed{\text{Step ①}}$

Example. $A = \mathbb{C}[\bar{x}]$, graded by degree.

$$\dim_{\mathbb{C}}(A_d) = \binom{m+d-1}{d} \qquad \left(\because \text{ spanned by } \left\{\prod_{i=1}^{m} x_i^{\alpha_i}\right\} \text{ with } \underline{\sum_{i=1}^{m} \alpha_i = d} .\right)$$

$$\Phi_A(z) = \sum_{d=0}^{\infty} \binom{m+d-1}{d} z^d = \frac{1}{(1-z)^m} .$$

• For (slightly) more general setting, similar result holds.

Lemma Let $p_1, \ldots, p_k$ be algebraically independent elements of $\mathbb{C}[x]$, homogeneous of degree $d_1, \ldots, d_k$.

Let $R := \mathbb{C}[p_1, \ldots, p_k]$. Then

$$\Phi_R(z) = \prod_{i=1}^{k} (1 - z^{d_i})^{-1}.$$

Proof. $\dim_{\mathbb{C}}(R_d)$: spanned by $\prod_{i=1}^{k} p_i^{\alpha_i}$, where $\sum \alpha_i \cdot d_i = d$.

$$\Phi_R(z) = \sum_{d \geq 0} \left(\sum_{\alpha : \sum \alpha_i \cdot d_i = d} 1 \right) \cdot z^d$$

$$= \prod_{i=1}^{k} \left(1 + z^{d_i} + z^{2d_i} + \ldots \right)$$

$$= \prod_{i=1}^{k} \frac{1}{1 - z^{d_i}}.$$

Generalizing to Cohen-Macaulay algebra.

- Expect similar things for Cohen-Macaulay algebra!

Cor Let A: graded Cohen-Macaulay $\mathbb{C}$-algebra, with Hironaka decomposition

$$A = \bigoplus_{i=1}^{t} \mathbb{C}[\theta_1, \dots, \theta_n] \cdot \eta_i .$$

Then $\boxed{\Phi_A(z) = \sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} (1 - z^{d_j})}$.

Step ②

where $d_j = \deg(\theta_j)$, $e_i = \deg(\eta_i)$.

Proof. Let $E_i := \mathbb{C}[\theta_1, \dots, \theta_n] \cdot \eta_i$.

$$\Phi_{E_i}(z) = z^{e_i} \Big/ \prod_{j=1}^{n} (1 - z^{d_j}) .$$

Since $A = \bigoplus_i E_i$, $\Phi_A(z) = \sum_{i=1}^{t} \Phi_{E_i}(z) = \sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} (1 - z^{d_j})$.

Molien's formula

Thm Denote by $\Phi_G(z)$ the Hilbert series of $\mathbb{C}[\bar{x}]^G$.

Then, $$\boxed{\Phi_G(z) = \frac{1}{|G|} \sum_{g \in G} \det(I_m - z\underbrace{g}_{m\times m \text{ matrix}})^{-1}.}$$

Step ③.

$$\Phi_G(z) = \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(\mathbb{C}_d[\bar{x}]^G) \cdot z^d.$$

- It is best to look at the "induced operator" acting on $\mathbb{C}^D$, with $D = \binom{m+d-1}{d} = \dim_{\mathbb{C}}(\mathbb{C}_d[\bar{x}])$.
- Let's see what happens if G simply acts on some $\mathbb{C}^D$.

Lemma The dimension of $(\mathbb{C}^D)^G$ is equal to

$$\frac{1}{|G|} \cdot \sum_{g \in G} \operatorname{tr}(g).$$

Proof. Consider averaging operator

$$R(x) := \frac{1}{|G|} \sum_{g \in G} (g x).$$

- $R : \mathbb{C}^D \to \mathbb{C}^D$ is a projection on $(\mathbb{C}^D)^G$.

$$\Rightarrow \dim((\mathbb{C}^D)^G) = \operatorname{tr}(R)$$

$$= \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g).$$

Proof of Molien's formula.

Consider $g \in G$. $\mathbb{C}_1[\bar{x}]$

Say it has eigenvectors $\ell_1^{(g)}, \dots, \ell_n^{(g)}$,

with corresponding e-values $\lambda_1^{(g)}, \dots, \lambda_n^{(g)}$.

$\Rightarrow$ g acting on $\mathbb{C}_d[\bar{x}]$ has:

eigenvectors $\left\{ \prod_{i=1}^{n} \left(\ell_i^{(g)}\right)^{\alpha_i} : \sum_{i=1}^{n} \alpha_i = d \right\}$.

eigenvalues $\prod_{i=1}^{n} \left(\lambda_i^{(g)}\right)^{\alpha_i} : \sum_{i=1}^{n} \alpha_i = d$.

$$\therefore \dim_{\mathbb{C}}\left(\mathbb{C}_d[\bar{x}]\right) = \frac{1}{|G|} \sum_{g \in G} \operatorname{tr}(g).$$

$$= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{|\alpha| = d} \prod_{i=1}^{n} \left(\lambda_i^{(g)}\right)^{\alpha_i} \right).$$

$$
\begin{aligned}
\Phi_G(z) &= \sum_{d\geq 0} \dim_{\mathbb{C}}\left(\mathbb{C}_d[\bar{x}]^G\right) z^d \\
&= \sum_{d\geq 0}\left[\frac{1}{|G|}\sum_{g\in G}\sum_{|\alpha|=d}\prod_{i=1}^{n}\left(\lambda_i^{(g)}\right)^{\alpha_i}\right] z^d \\
&= \frac{1}{|G|}\sum_{g\in G}\left[\sum_{d\geq 0}\sum_{|\alpha|=d}\prod_{i=1}^{n}\left(z\lambda_i^{(g)}\right)^{\alpha_i}\right] \\
&= \frac{1}{|G|}\sum_{g\in G}\left[\sum_{\alpha\geq 0}\prod_{i=1}^{n}\left(z\lambda_i^{(g)}\right)^{\alpha_i}\right] \\
&= \frac{1}{|G|}\sum_{g\in G}\left(\prod_{i=1}^{n}\frac{1}{1-z\lambda_i^{(g)}}\right) \\
&= \frac{1}{|G|}\sum_{g\in G}\det\left(I_n - zg\right)^{-1}.
\end{aligned}
$$

$\square$

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Putting everything together.

- Let $\mathbb{C}[\bar{x}]^G = \bigoplus_{i=1}^{t} \mathbb{C}[\theta_1, \dots, \theta_n] \cdot \eta_i$; $\deg(\theta_j) = d_j$, $\deg(\eta_i) = e_i$.
- By Step ②, $\Phi_G(z) = \sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} (1 - z^{d_j})$.
- By Step ③, $\Phi_G(z) = \frac{1}{|G|} \cdot \sum_{g \in G} \det(I_n - zg)^{-1}$.

$$\Rightarrow \quad \frac{1}{|G|} \cdot \sum_{g \in G} \det(I_n - zg)^{-1} = \frac{\sum_{i=1}^{t} z^{e_i}}{\prod_{j=1}^{n} (1 - z^{d_j})}. \qquad \text{--- } (*)$$

$\boxed{\text{Step ④}}$

Thm (a) $\sum_{i=1}^{t} z^{e_i} = \Phi_G(z) \cdot \prod_{j=1}^{n} (1-z^{d_j})$.

(b) $t = \dfrac{\prod_{j=1}^{n} d_j}{|G|}$.

Proof. (a) $\sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} (1-z^{d_j}) = \Phi_G(z)$.

(b) $\frac{1}{|G|} \sum_{g\in G} \det(I_n - zg)^{-1} = \sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} (1-z^{d_j})$.

Multiply $(1-z)^n$ on both sides,

$$\frac{1}{|G|} \sum_{g\in G} \frac{(1-z)^n}{\det(I_n - zg)} = \sum_{i=1}^{t} z^{e_i} \Big/ \prod_{j=1}^{n} \left(\frac{1-z^{d_j}}{1-z}\right)$$

Let $z \to 1$, LHS $= \frac{1}{|G|}$.

RHS $= t \Big/ \prod_{j=1}^{n} d_j$. Rearrange $\Rightarrow t = \dfrac{\prod_{j=1}^{n} d_j}{|G|}$.

Making use of this arithmetic information: 1st attempt

If G is explicitly given:

① Compute $\Phi_G(z)$ using Molien's formula: $\Phi_G(z) = \frac{1}{|G|} \sum_{g \in G} \det(I_n - zg)^{-1}$.

② Write it as a rational function, match it with

$$\frac{\sum_{i=1}^{t} z^{e_i}}{\prod_{j=1}^{n} (1 - z^{d_j})}.$$

③ $(d_1, \dots, d_n)$ gives a good guess of the degrees of the sought primary invariants $\theta_1, \dots, \theta_n$.

④ Use Reynolds operator to find $\theta_1, \dots, \theta_n$, also $\eta_1, \dots, \eta_t$.

Example $G = \mathbb{Z}_4$, represented by $\left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$

Find Hironaka decomposition of ~~$\mathbb{Q}$~~ $\mathbb{C}[x,y]^G$.

$$\Phi_G(z) = \frac{1}{4}\left[\det\left(\begin{pmatrix} 1-z & 0 \\ 0 & 1-z \end{pmatrix}\right)^{-1} + \det\left(\begin{pmatrix} 1+z & 0 \\ 0 & 1+z \end{pmatrix}\right)^{-1} + \det\left(\begin{pmatrix} 1 & z \\ -z & 1 \end{pmatrix}\right)^{-1} + \det\left(\begin{pmatrix} 1 & -z \\ z & 1 \end{pmatrix}\right)^{-1} \right]$$

$$= \frac{1}{4}\left[\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2} + \frac{2}{1+z^2} \right]$$

$$= \frac{1}{4}\left[\frac{(1+z)^2(1+z^2) + (1-z)^2(1+z^2) + 2(1-z)^2(1+z)^2}{(1-z)^2(1+z)^2(1+z^2)} \right]$$

$$= \frac{1+z^4}{(1-z^2)(1-z^4)}$$

$R(x^4) = \frac{1}{2}(x^4+y^4)$

$R(x^2) = \frac{1}{2}(x^2+y^2)$

$R(x^2y^2) = x^2y^2$ . .

$R(x^3y) = \frac{1}{2}(x^3y - xy^3)$

Algorithm for computing a Hironaka decomposition of $\mathbb{C}[\bar{x}]^G$.

① Compute $\Phi_G(z)$ using Molien's formula

② Find a h.s.o.p. $\{\theta_1, \dots, \theta_n\}$ (using e.g. Dade's algorithm)

③ To find secondary invariants, write

$$\Phi_G(z) \cdot \prod_{j=1}^{n} (1 - z^{d_j}) = \sum_i c_i \cdot z^{e_i} \quad , \; c_i \in \mathbb{Z}^+ .$$

④ Apply averaging operator $R(\bar{x}) := \frac{1}{|G|} \sum_{g \in G} g\bar{x}$;

$\rightsquigarrow$ find invariants of degree e_i that are linearly independent modulo ideal generated by $\theta_1, \dots, \theta_n$.

* Details please see Section 2.5 .

Summary

- Introduced Hilbert series
- Proved a relation between primary & secondary invariants of $\mathbb{C}[\bar{x}]^G$
- Looked at how the relation helps compute Hironaka decomposition.

Reference

- Sects. 2.2 & 2.3 of "Algorithms in Invariant Theory"