Lecture 23: General Lax Conjecture & its Variants

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science
rafael.oliveira.teaching@gmail.com

April 6, 2021
Overview

- General Lax Conjecture and Variants
- Conditional Lower Bounds on Spectrahedral Representation
- Conclusion
- Acknowledgements
Hyperbolic Programming

Definition (Hyperbolic Programming)

Given $h(x) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic with respect to $e \in \mathbb{R}^m$, a hyperbolic program is the following minimization problem:

$$\inf \ c^\top x$$

s.t. $x \in \Lambda_+(h, e)$

Remark

Hyperbolic programming generalizes Linear Programming (LP) and Semidefinite Programming (SDP)!

Hyperbolic programming with $h(x) = \ell_1(x) \cdots \ell_m(x)$ gives rise to LPs.

Hyperbolic programming with $h(x) = \det(\sum A_i x_i)$, with $A_i$ symmetric matrices gives rise to SDPs.
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**Spectrahedral Sets & SDPs**

**Definition (Spectrahedral Sets)**

A convex set \( S \subseteq \mathbb{R}^m \) is **spectrahedral** if it can be defined by linear matrix inequalities (LMIs). That is, there exists \( d \in \mathbb{N} \) and \( d \times d \) symmetric matrices \( A_1, \ldots, A_m, B \) such that

\[
S = \{ c \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq B \}.
\]

\( S \) has non-empty interior if there is \( e \in S \) such that \( \sum_i e_i \cdot A_i \succ B \).

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$S$ has non-empty interior if there is $e \in S$ such that $\sum_i e_i \cdot A_i \succ B$.

**Open Question (General Lax Conjecture)**

Is every hyperbolicity cone a spectrahedral set?

This question relates the qualitative generality of Hyperbolic Programming compared with SDPs.

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General Lax Conjecture

\[ LP \subset SDP \subset HP. \]

- First containment proper.

Original conjecture was only for hyperbolic polynomials in 3 variables, which was proved by [Helton, Vinnikov, 2007]

General Lax Conjecture about qualitative aspects of SDPs vs HPs. Can we get quantitative aspects between them?

Open Question (Quantitative General Lax Conjecture)

Is there a hyperbolicity cone which is "simple", but any spectrahedral representation of it requires matrices of large dimension?

Open Question (Explicit "hard" hyperbolicity cone)

Is there an explicit hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?
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- **General Lax Conjecture**: the last containment is in fact an *equality*

\[ SDP = HP \quad (\text{qualitatively}) \]

\[
\inf C^T x
\]

\[ \text{subject to } x \in \Lambda_f(h, \overline{e}) \]

\[ \sum_{i=1}^{m} A_i x_i \geq 0 \]

\[ \sum A_i e_i > 0 \text{ in int}\Lambda_f \]
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- Is there a hyperbolicity cone which is "simple", but any spectrahedral representation of it requires matrices of large dimension?
- Is there an explicit hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?

\[ \Lambda_i(h, \bar{e}) \leftrightarrow \sum A_i x_i \geq 0 \]

- Small description?
- Don't need bounds on \( \dim(A_i) \)
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*Is there an explicit hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?*
Open Question (Quantitative Approximate General Lax Conjecture)

Is there an explicit hyperbolicity cone for which any approximate spectrahedral representation of it requires matrices of super polynomial dimension?

\[
\inf c^*x \\
\text{s.t. } \sum A_i x_i \leq 0 \\
K \preceq_\delta K_\delta
\]
### Variants of General Lax Conjecture

**Open Question (Quantitative Approximate General Lax Conjecture)**

*Is there an *explicit* hyperbolicity cone for which any *approximate* spectrahedral representation of it requires matrices of super polynomial dimension?*

**Open Question (Projected Lax Conjecture)**

*Can all hyperbolicity cones be represented as *spectrahedral shadows*?*

---

*Can make this Projected Lax conjecture question quantitative:*
Variants of General Lax Conjecture

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Can all hyperbolicity cones be represented as spectrahedral shadows?

Open Question (Extended Formulations)

Is there an explicit hyperbolicity cone for which any spectrahedral shadow representation of it requires matrices of super polynomial dimension? Question is open even for non-explicit polynomials.
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Open Question (Extended Formulations)

Is there an explicit hyperbolicity cone for which any spectrahedral shadow representation of it requires matrices of super polynomial dimension?
Question is open even for non-explicit polynomials.

And many more... this is just the beginning of the rabbit hole.
Previous Work

Theorem (Non-Explicit Lower Bounds [RRSW 2019])

**Exponential** lower bounds on the dimension of minimal spectrahedral representations of non-explicit hyperbolicity cones (which are known to be spectrahedral).

- Exponential lower bounds for some polynomial in a large set of hyperbolic polynomials
- Carefully chosen perturbations of elementary symmetric polynomial
Theorem (Non-Explicit Lower Bounds [RRSW 2019])

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Theorem (Explicit Linear Lower Bounds [Kummer 2016])

**Optimal** lower bounds on the dimension of minimal spectrahedral representations of **explicit** hyperbolicity cones of quadratic polynomials.

- Linear lower bounds (on number of variables) for Lorentz cone
- Matches upper bounds for known constructions
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Today: first (conditional) *superpoly* lower bound for *explicit* polynomials.
Previous Work

Definition (Smooth Hyperbolicity Cones)

A hyperbolicity cone $\Lambda_+(h, e)$ is smooth if each non-zero point in the boundary of $\Lambda_+(h, e)$ is a smooth point\(^a\) of $h$.

\(^a\)The point is not a singular point of $h$. 

Theorem (Smooth Hyperbolicity Cones [Netzer, Sanyal 2015])

Smooth hyperbolicity cones are spectrahedral shadows.

Smooth hyperbolic polynomials are dense over the set of all hyperbolic polynomials.

Projected Lax conjecture true “for most points”

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General Lax Conjecture and Variants

Conditional Lower Bounds on Spectrahedral Representation

Conclusion

Acknowledgements
General Lax Conjecture - Equivalent Formulation

$h(x) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic w.r.t. $e \in \mathbb{R}^m$, does there exist $d \in \mathbb{N}$ and symmetric $d \times d$ matrices $A_1, \ldots, A_m$ such that

$$\Lambda_+(h, e) = \{ c \in \mathbb{R}^m \mid \sum c_i \cdot A_i \succeq 0 \}$$

$LMI$

$$\sum A_i e_i \succ 0$$

(with nonempty interior)
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Definition (Definite Determinantal Representations)

A homogeneous polynomial $h(x) \in \mathbb{R}[x]$ has a \textit{definite determinantal representation} at $e \in \mathbb{R}^m$ if there are symmetric matrices $A_1, \ldots, A_m$ s.t.:

- $\sum_i e_i \cdot A_i \succeq 0$
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- $\sum_i e_i \cdot A_i \succeq 0$
- $h(x) = \det(\sum_i x_i \cdot A_i)$

Proposition (General Lax Conjecture - Equivalent Formulation)

Each hyperbolic polynomial $h(x)$ at $e \in \mathbb{R}^m$ can be multiplied by another hyperbolic polynomial $q(x)$ at $e$, such that $\Lambda_+(h, e) \subseteq \Lambda_+(q, e)$ and the polynomial $h(x) \cdot q(x)$ has a definite determinantial representation.
Minimal Defining Polynomials

A set $C \subset \mathbb{R}^m$ is an *algebraic interior* if there is a polynomial $p(x) \in \mathbb{R}[x]$ such that $C$ is the closure of a connected component of

$$\{a \in \mathbb{R}^m \mid p(a) > 0\}$$

$p$ is called a *defining polynomial* of $C$.
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- If $C$ is an algebraic interior, then a minimal degree polynomial defining $C$ is **unique** (up to units) **minimal defining polynomial** of $C$

- If $p$ is the minimal defining polynomial of $C$, any other defining polynomial $q$ of $C$ must be a **multiple** of $p$ in the following way:

$$q(x) = p(x) \cdot h(x)$$

where $h$ is **strictly positive** on a **dense connected subset** of $C$

$$\Lambda_+(h, \bar{e}) \supset \Lambda_+(p, \bar{e})$$
Factoring and Circuit Size

Theorem (Factors are closed in VP [Kaltofen 1989])

If a polynomial is in VP (i.e. has polynomial degree in the number of variables and can be computed by poly-sized algebraic circuits), then so do all of its factors.
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Corollary (Factors are closed in VP [Kaltofen 1989])

If a polynomial is not in VP, then no multiple of this polynomial is in VP either.
Main Result: Conditional Lower Bounds

Definition (Matching Polynomial [Amini 2019])

Let $G(V, E)$ be an undirected graph $x = (x_v)_{v \in V}$, $w = (w_e)_{e \in E}$ be indeterminates.

- $\mathcal{M}(G)$ be the set of all matchings of $G$, $\mathcal{M}(G) \subseteq 2^E$
- for $M \in \mathcal{M}(G)$ let $V(M)$ be the vertices in this matching

$$
\mu_G(x, w) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \cdot \prod_{v \notin V(M)} x_v \cdot \prod_{e \in M} w_e^2.
$$

Amini showed that this polynomial is hyperbolic and the hyperbolicity cone of $\mu_G$ is spectrahedral.

Amini constructed spectrahedral representation of dimension $n!$. 
Main Result: Conditional Lower Bounds

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$$\mathcal{C} = (\mathcal{F}^v, \mathcal{O}_E)$$

Theorem (Lower Bounds for Spectrahedral Representations)

If $G = K_{n,n}$ is the complete bipartite graph, then the minimal spectrahedral representation of the hyperbolicity cone of $\mu_G$ is superpolynomial, assuming that VP $\neq$ VNP.
Proof strategy: \( (\Lambda = \Lambda_{\text{min}}, \varepsilon) \)

1) \( \Lambda \) is VNP-hard

2) \( \mu \) is irreducible
   \[ \Rightarrow \mu \text{ is minimal defining polynomial of its hyperbolicity cone w.r.t. } \varepsilon \]

3) Kaltofen's result to show that any definite determinantal representation must be large.
VNP-hardness of Matching Polynomial

\[ M(x, \bar{w}) = \sum_{M \in \mathcal{M}} (-1)^{\mu_M} \prod_{(u,v) \in \mathcal{M}} x_{uv} \prod_{e \notin M} \bar{w}_e^2 \]

\[ M(\bar{0}, \bar{w}) = (-1)^n \sum_{\sigma \in S_n} \prod_{i=1}^{n} \bar{w}_{\sigma(i)} \]

\[ = (-1)^n \cdot \text{Per}(W) \]

\[ W_{ij} = \bar{w}_{ij} \]

\[ M(\bar{0}, \sqrt{1-\bar{w}}) = (-1)^n \text{Per}(J-U) \]

from here prove that if \( \exists \Phi \in \text{VP} \) then computing \( \Psi \in \text{VP} \)
VNP-hardness of Matching Polynomial
VNP-hardness of Matching Polynomial
Irreducibility of Matching Polynomial

\[ M(x, \bar{w}) = \sum_{\text{Mon}} (-1)^{\text{Mon}} \prod_{\text{odd} \neq M} x_o \prod_{\text{even} M} \bar{w}_o^2 \]

\[ M = p(x, \bar{w}) \cdot q(x, \bar{w}) \]

\[ M(\bar{0}, \bar{w}) = \prod_{\text{Perm}} p(\bar{0}, \bar{w}) \cdot q(\bar{0}, \bar{w}) \]

\[ (-1)^{\text{Perm}} \cdot \text{Perm}(\bar{w}) \]

\[ \text{inreducible} \]

\[ M(\bar{x}, \bar{0}) = \prod_{v \in M} \bar{x}_v \]

\[ \text{"monic"} \]
Proof: Factoring Implies Multiples are Hard too

\[ \det \left( \sum A_i x_i + \sum B e w e \right) = M \cdot q(x, \tilde{w}) \]

spectrahedral representation

1) \( \dim(A_i) = \dim(B_e) \) super poly
   done

2) \( \dim(A_i) = \dim(B_e) = \Pi^C \)
   \[ \Rightarrow \det(\sum A_i x_i + \sum B e w e) \in \text{UP} \]
   \[ \text{and multiple of \text{VNP}-hard} \ M \]

\[ \text{VP} \neq \text{VNP} \]
contradiction
Conclusion

- Today we learned about the general Lax conjecture and their variants, including computational ones!
- All of them are open
- Connections to algebraic complexity, convex algebraic geometry, real algebraic geometry and real-stability!
Acknowledgement

- Lecture based largely on [Oliveira 2020]
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