Lecture 23: General Lax Conjecture & its Variants

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Overview

- General Lax Conjecture and Variants
- Conditional Lower Bounds on Spectrahedral Representation
- Conclusion
- Acknowledgements

Hyperbolic Programming

Definition (Hyperbolic Programming)

Given $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^m$, a hyperbolic program is the following minimization problem:

$$\begin{aligned} &\inf\ \mathbf{c}^\dagger \mathbf{x} \\ &\text{s.t. } \mathbf{x} \in \Lambda_+(h, \mathbf{e}) \end{aligned}$$

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Remark

Hyperbolic programming generalizes Linear Programming (LP) and Semidefinite Programming (SDP)!

- Hyperbolic programming with $h(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_m(\mathbf{x})$ gives rise to LPs
- Hyperbolic programming with $h(\mathbf{x}) = \det(\sum A_i x_i)$, with A_i symmetric matrices gives rise to SDPs

Spectrahedral Sets & SDPs[1]

Definition (Spectrahedral Sets)

A convex set $S \subseteq \mathbb{R}^m$ is spectrahedral if it can be defined by linear matrix inequalities (LMIs). That is, there exists $d \in \mathbb{N}$ and $d \times d$ symmetric matrices $A_1, \ldots, A_m, B$ such that

$$S = \{\mathbf{c} \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq B\}.$$

S has non-empty interior if there is $\mathbf{e} \in S$ such that $\sum_i e_i \cdot A_i \succ B$.

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Open Question (General Lax Conjecture)

Is every hyperbolicity cone a spectrahedral set?

This question relates the qualitative generality of Hyperbolic Programming compared with SDPs.

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General Lax Conjecture

$$LP \subset SDP \subseteq HP.$$

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$SDP = HP$ (qualitatively)

$\inf\ c^T x$

s.t. $x \in \Lambda_+(h, \bar{e})$

$\sum_{i=1}^{m} A_i x_i \succeq 0$

$\sum A_i e_i \succ 0$ in interior

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- General Lax Conjecture about *qualitative* aspects of SDPs vs HPs. Can we get *quantitative* aspects between them?

small description?

$$\Lambda_+(h, \bar{e}) \longleftrightarrow \sum A_i x_i \succeq 0$$

small description

$$\sum A_i e_i \succ 0$$

don't need bounds on $\dim(A_i)$

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Open Question (Quantitative General Lax Conjecture)

Is there a hyperbolicity cone which is "**simple**"*, but any spectrahedral representation of it requires matrices of large dimension?*

simple: poly-sized algebraic circuits

→ super-poly size

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Open Question (Quantitative General Lax Conjecture)

Is there a hyperbolicity cone which is **"simple"***, but any spectrahedral representation of it requires matrices of large dimension?*

Open Question (Explicit "hard" hyperbolicity cone)

Is there an explicit hyperbolicity cone for which any spectrahedral representation of it requires matrices of large dimension?

Variants of General Lax Conjecture

Open Question (Quantitative Approximate General Lax Conjecture)

Is there an explicit hyperbolicity cone for which any **approximate** *spectrahedral representation of it requires matrices of super polynomial dimension?*

approx. $\varepsilon > 0$

$$\inf \; c^T x$$

$$\text{s.t. } \underbrace{\sum A_i x_i \succeq 0}$$

$$K \approx_\delta K_\delta$$

Variants of General Lax Conjecture

Open Question (Quantitative Approximate General Lax Conjecture)

Is there an explicit hyperbolicity cone for which any **approximate** *spectrahedral representation of it requires matrices of super polynomial dimension?*

Open Question (Projected Lax Conjecture)

Can all hyperbolicity cones be represented as spectrahedral shadows?

Can make this Projected Lax conjecture question quantitative?

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Is there an ***explicit*** *hyperbolicity cone for which any spectrahedral shadow representation of it requires matrices of super polynomial dimension? Question is open even for non-explicit polynomials.*

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And many more... this is just the beginning of the rabbit hole.

Previous Work

Theorem (Non-Explicit Lower Bounds [RRSW 2019])

Exponential *lower bounds on the dimension of minimal spectrahedral representations of* **non-explicit** *hyperbolicity cones (which are known to be spectrahedral).*

- *Exponential lower bounds for some polynomial in a large set of hyperbolic polynomials*
- *Carefully chosen perturbations of* **elementary symmetric polynomial**

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Theorem (Explicit Linear Lower Bounds [Kummer 2016])

Optimal *lower bounds on the dimension of minimal spectrahedral representations of* **explicit** *hyperbolicity cones of quadratic polynomials.*

- **Linear** *lower bounds (on number of variables) for Lorentz cone*
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Today: first (conditional) *superpoly* lower bound for **explicit** polynomials.

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Definition (Smooth Hyperbolicity Cones)

A hyperbolicity cone $\Lambda_+(h, \mathbf{e})$ is *smooth* if each non-zero point in the boundary of $\Lambda_+(h, \mathbf{e})$ is a *smooth point*[a] of h.

[a] The point is not a singular point of h.

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Theorem (Smooth Hyperbolicity Cones [Netzer, Sanyal 2015])

Smooth hyperbolicity cones are spectrahedral shadows.

- *Smooth hyperbolic polynomials are dense over the set of all hyperbolic polynomials.*
- *Projected Lax conjecture true "for most points"*

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General Lax Conjecture - Equivalent Formulation

$h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_m]$ hyperbolic w.r.t. $\mathbf{e} \in \mathbb{R}^m$, does there exist $d \in \mathbb{N}$ and symmetric $d \times d$ matrices $A_1, \dots, A_m$ such that

$$\Lambda_+(h, \mathbf{e}) = \{\mathbf{c} \in \mathbb{R}^m \mid \sum_i c_i \cdot A_i \succeq 0\}$$

LMI

$$\sum A_i e_i \succ 0$$

(with nonempty interior)

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Definition (Definite Determinantal Representations)

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ has a *definite determinantal representation* at $\mathbf{e} \in \mathbb{R}^m$ if there are symmetric matrices $A_1, \ldots, A_m$ s.t.:

- $\sum_i e_i \cdot A_i \succ 0$
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Proposition (General Lax Conjecture - Equivalent Formulation)

Each hyperbolic polynomial $h(\mathbf{x})$ at $\mathbf{e} \in \mathbb{R}^m$ can be multiplied by another hyperbolic polynomial $q(\mathbf{x})$ at $\mathbf{e}$, such that $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(q, \mathbf{e})$ and the polynomial $h(\mathbf{x}) \cdot q(\mathbf{x})$ has a definite determinantal representation.

Minimal Defining Polynomials

- A set $\mathcal{C} \subset \mathbb{R}^m$ is an *algebraic interior* if there is a polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ such that $\mathcal{C}$ is the closure of a connected component of

$$\{\mathbf{a} \in \mathbb{R}^m \ \mid \ p(\mathbf{a}) > 0\}$$

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  p is called a *defining polynomial* of $\mathcal{C}$
- If $\mathcal{C}$ is an algebraic interior, then a minimal degree polynomial defining $\mathcal{C}$ is *unique* (up to units) *minimal defining polynomial* of $\mathcal{C}$
- If p is the minimal defining polynomial of $\mathcal{C}$, any other defining polynomial q of $\mathcal{C}$ must be a *multiple* of p in the following way:

$\det(\sum x_i A_i)$

$$q(\mathbf{x}) = p(\mathbf{x}) \cdot h(\mathbf{x})$$

  where h is *strictly positive* on a *dense connected subset* of $\mathcal{C}$

$\Lambda_+(h, \bar{e}) \supset \Lambda_+(p, \bar{e})$

Factoring and Circuit Size

Theorem (Factors are closed in VP [Kaltofen 1989])

If a polynomial is in VP (i.e. has polynomial degree in the number of variables and can be computed by poly-sized algebraic circuits), then so do all of its factors.

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"Corollary" (Factors are closed in VP [Kaltofen 1989])

If a polynomial is not in VP, then no multiple of this polynomial is in VP either.

Main Result: Conditional Lower Bounds

Definition (Matching Polynomial [Amini 2019])

Let $G(V, E)$ be an undirected graph $\mathbf{x} = (x_v)_{v \in V}$, $\mathbf{w} = (w_e)_{e \in E}$ be indeterminates.

- $\mathcal{M}(G)$ be the set of all matchings of G, $\mathcal{M}(G) \subseteq 2^E$
- for $M \in \mathcal{M}(G)$ let $V(M)$ be the vertices in this matching

$$\mu_G(\mathbf{x}, \mathbf{w}) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \cdot \prod_{v \notin V(M)} x_v \cdot \prod_{e \in M} w_e^2.$$

Amini showed that this polynomial is hyperbolic and the hyperbolicity cone of μ_G is spectrahedral.

Amini constructed spectrahedral representation of dimension $n!$

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Amini showed that this polynomial is hyperbolic and the hyperbolicity cone of μ_G is spectrahedral. $\bar{e} = (1_V, 0_E)$

Theorem (Lower Bounds for Spectrahedral Representations)

If $G = K_{n,n}$ is the complete bipartite graph, then the minimal spectrahedral representation of the hyperbolicity cone of μ_G is superpolynomial, assuming that $VP \neq VNP$.

Proof strategy: $(\mathcal{M} = \mathcal{M}_{n,n}, \bar{e})$

1) $\mathcal{M}$ is VNP-hard

2) $\mathcal{M}$ is irreducible

($\Rightarrow$ $\mathcal{M}$ is minimal defining polynomial of its hyperbolicity cone w.r.t. $\bar{e}$)

3) Kaltofen's result to show that any definite determinantal representation must be large.

VNP-hardness of Matching Polynomial

$G = K_{n,n}$

$$\mathcal{M}(\bar{x}, \bar{w}) = \sum_{M \in \mathcal{M}} (-1)^{|M|} \prod_{v \notin V(M)} x_v \prod_{e \in M} w_e^2$$

$$\mathcal{M}(\bar{0}, \bar{w}) = (-1)^n \sum_{\sigma \in S_n} \prod_{i=1}^{n} w_{i\sigma(i)}^2$$

$$= (-1)^n \cdot \mathrm{Per}(W)$$

$$W_{ij} = w_{ij}^2$$

$$\mathcal{M}(\bar{0}, \sqrt{1-\bar{u}}) = (-1)^n \, \mathrm{Per}(J - U)$$

from here prove that if $\exists\, \Phi \in VP$ computing $\mathcal{M}$ then $\Psi \in VP$ computing Per.

VNP-hardness of Matching Polynomial

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Irreducibility of Matching Polynomial

$$\mu(\bar{x}, \bar{w}) = \sum_{M \in \mathcal{M}} (-1)^{|M|} \prod_{v \notin V(M)} x_v \prod_{e \in M} w_e^2$$

$$\mu = p(\bar{x}, \bar{w}) \cdot q(\bar{x}, \bar{w})$$

$x_1 x_2$

not perm

$$\mu(\bar{0}, \bar{w}) = \boxed{p(\bar{0}, \bar{w})} \cdot q(\bar{0}, \bar{w})$$

$(-1)^n \cdot \mathrm{Per}(w^2)$

irreducible

Perm

unit

$x_1 x_2$

$x_1 x_2$

$$\mu(\bar{x}, \bar{0}) = \prod_{v \notin M} x_v$$

("monic")

Proof: Factoring Implies Multiples are Hard too

$$\det\left(\sum A_i x_i + \sum B_e w_e\right) = f \cdot q(\bar{x}, \bar{w})$$

spectrahedral representation

1) $\dim(A_i) = \dim(B_e)$ superpoly

done

2) $\dim(A_i) = \dim(B_e) = n^c$

$\Rightarrow \det\left(\sum A_i x_i + \sum B_e w_e\right) \in VP$

and multiple of VNP-hard f

$\therefore$ $VP \neq VNP$ contradiction

Conclusion

- Today we learned about the general Lax conjecture and their variants, including computational ones!
- All of them are open
- Connections to algebraic complexity, convex algebraic geometry, real algebraic geometry and real-stability!

Acknowledgement

- Lecture based largely on [Oliveira 2020]

References I

Helton, J William and Vinnikov, Victor (2007)
Linear matrix inequality representation of sets
Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 60(5), 654–674.

Kaltofen, Erich (1989)
Factorization of polynomials given by straight-line programs
Randomness and Computation 5, 375–412

Kummer, Mario (2016)
Two results on the size of spectrahedral descriptions
SIAM Journal on Optimization, 26(1), 589 – 601.

Raghavendra, Prasad and Ryder, Nick and Srivastava, Nikhil and Weitz, Benjamin (2019)
Exponential lower bounds on spectrahedral representations of hyperbolicity cones
Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms 2322 – 2332

References II

Netzer, Tim, and Raman Sanyal (2015)
Smooth hyperbolicity cones are spectrahedral shadows.
Mathematical Programming 153, no. 1 (2015): 213-221.

Amini, Nima (2019)
Spectrahedrality of hyperbolicity cones of multivariate matching polynomials
Journal of Algebraic Combinatorics 50(2), 165 – 190

Oliveira, Rafael (2020)
Conditional Lower Bounds on the Spectrahedral Representation of Explicit Hyperbolicity Cones
45th International Symposium on Symbolic and Algebraic Computation (ICALP)