Lecture 22: Spectrahedral & Semidefinite Representations

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April 5, 2021

Overview

- Administrivia
- Important Hyperbolic Polynomials and Complexity of Verifying Hyperbolicity
- Spectrahedral Sets and Semidefinite Representations

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Please provide us (and the school) with your evaluation and feedback on the course!

- This would really help me figuring out what worked and what didn't for the course
- And let the school know if I was a good boy this term!
- Teaching this course is also a learning experience for me :)

- Administrivia

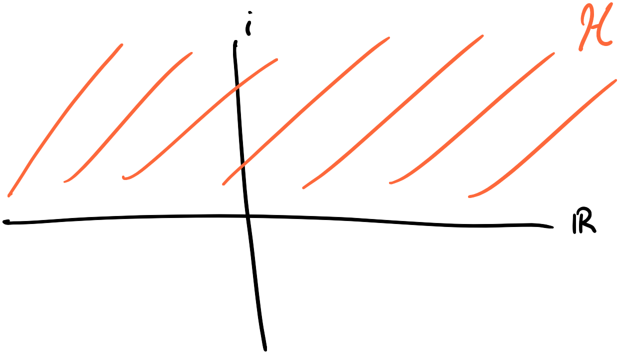
- Important Hyperbolic Polynomials and Complexity of Verifying Hyperbolicity

- Spectrahedral Sets and Semidefinite Representations

Real-Stable Polynomials

- Let $\mathcal{H}^m \subset \mathbb{C}^m$ be the complex upper-half region, where

$$\mathcal{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$$



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- A polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is called real-stable iff it is the zero polynomial or it has *no zeros* in the upper-half region of $\mathbb{C}^m$

$p(\bar{a} + i\bar{b}) \neq 0$

$\bar{b} > \bar{0}$ $\quad$ $\bar{a} \in \mathbb{R}^m$

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- Examples:
  1. Determinant of mixture of PSD matrices

$$p(\mathbf{x}) = \det(x_1 A_1 + \cdots + x_m A_m + B)$$

  where A_i's are PSD and B is Hermitian.

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- A polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is real stable iff for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} > 0$, the univariate polynomial $p(\mathbf{a} + \mathbf{b}t)$ only has real roots

real zeros polynomials ↔ "hyperbolic on affine space"

Real Stability and Hyperbolicity

- Let $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial. Then

$$p(\mathbf{x}) \text{ is real-stable iff } p(\mathbf{x}) \text{ is hyperbolic w.r.t. } \mathbf{e} = (1, \ldots, 1) \text{ and } \mathbb{R}_+^m \subset \Lambda_{++}(p, \mathbf{e})$$

positive orthant

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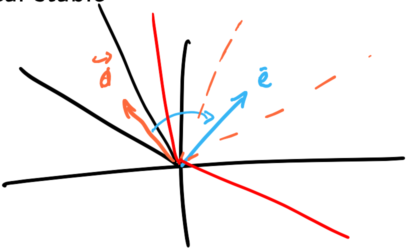
- Up to a linear change of coordinates every hyperbolic polynomial can be made real stable



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- Corollary of the lemma in previous slide.
- Up to a linear change of coordinates every hyperbolic polynomial can be made real stable
- Real-stable polynomials have many nice properties, so approaching hyperbolic questions via real stability can be quite useful

Matching Polynomials

- Given a graph $G(V,E)$ the (homogeneous multivariate) *matching polynomial*

$$\mu_G(\mathbf{x},\mathbf{w}) = \sum_{M \in \mathcal{M}(G)} (-1)^{|M|} \prod_{u \notin V(M)} x_u \cdot \prod_{e \in M} w_e^2$$

is hyperbolic w.r.t. $\mathbf{e} = (1_V, \mathbf{0})$.

edges in matching

not covered by matching

e is edge of matching

all matchings

(not only perfect ones)
include partial matchings

$\bar{x}_V$ $\bar{w}_E$

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- Hyperbolicity follows from the Heilmann-Lieb theorem.
- Another way to see it is via stability-preserving maps (but also require more background on stable polynomials)

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$$p(x_0, \mathbf{x}) = x_0^3 - 3x_0\underbrace{(x_1^2 + \cdots + x_n^2)}_{\|\bar{x}\|_2^2} + 2q(\mathbf{x})$$

for some $q(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]_3$ and $\mathbf{e} = (1, \mathbf{0})$.

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- By a theorem of Nesterov, if $G(V, E)$ is a simple graph and

$$q_G(\mathbf{x}, \mathbf{y}) = \sum_{\{i,j\} \in E} x_i x_j y_{i,j}$$

then

$$\max_{\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = 1} q_G(\mathbf{x}, \mathbf{y}) = \sqrt{2/27} \cdot \sqrt{1 - \frac{1}{\omega(G)}}$$

where $\omega(G)$ is the size of the largest clique in the graph G.

Complexity of Verifying Hyperbolicity

- Putting it all together, we have that

$$p_{G,k}(x_0, \mathbf{x}, \mathbf{y}) = \frac{2k}{k-1} \cdot x_0^3 - x_0(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) + q_G(\mathbf{x}, \mathbf{y})$$

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  is hyperbolic with respect to $\mathbf{e} = (1, \mathbf{0})$ iff $\omega(G) \leq k$.
- This implies that verifying hyperbolicity is *co-NP hard*
- This result is only for cubics... how hard can this problem really be?

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- Spectrahedral Sets and Semidefinite Representations

Spectrahedra

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Definition (Linear Matrix Inequalities)

A linear matrix inequality is an inequality of the form:

$$A_0 + \sum_{i=1}^{n} A_i x_i \succeq 0,$$

↑ Löwner order

where $A_0, \ldots, A_n$ are *symmetric matrices*.

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Definition (Spectrahedron)

A spectrahedron is a set defined by finitely many LMIs. In other words, it can be defined as:

$$S = \left\{ x \in \mathbb{R}^n \ \middle|\ \sum_{i=1}^{n} A_i x_i \succeq B, \quad A_i, B \in \mathcal{S}^m \right\}$$

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$$S = \left\{ \bar{x} \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \sum A_i x_i \succeq B_1 \\ \sum C_i x_i \succeq B_2 \end{array} \right\}$$

$$D_i = \begin{pmatrix} A_i & \\ & C_i \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix}$$

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$\bar{x}, \bar{y} \in S \qquad \alpha x + (1-\alpha) y$

$$\sum A_i (\alpha x_i + (1-\alpha) y_i) = \alpha \sum A_i x_i + (1-\alpha) \sum A_i y_i$$

$$\succeq \alpha B + (1-\alpha) B = B$$

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Example of Spectrahedron

$LP \subset SDP$

Polyhedron:

$$P = \{ x \in \mathbb{R}^n \mid Ax \geq b \}$$

$$\sum_{i=1}^{n} A_{ki} x_i \geq b_i$$

$$\sum x_i \begin{pmatrix} A_{1i} & & & \\ & A_{2i} & & \\ & & \ddots & \\ & & & A_{mi} \end{pmatrix} \succeq \begin{pmatrix} b_1 & & & \\ & b_2 & & \\ & & \ddots & \\ & & & b_m \end{pmatrix}$$

Example of Spectrahedron

Circle:

$$\mathcal{C} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

$$\Updownarrow$$

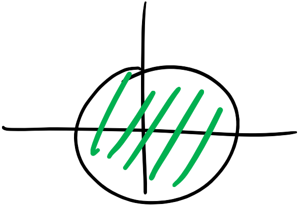
$$\begin{pmatrix} 1+x & y \\ y & 1-x \end{pmatrix} \succeq 0$$

iff

$$1 + x \geq 0$$

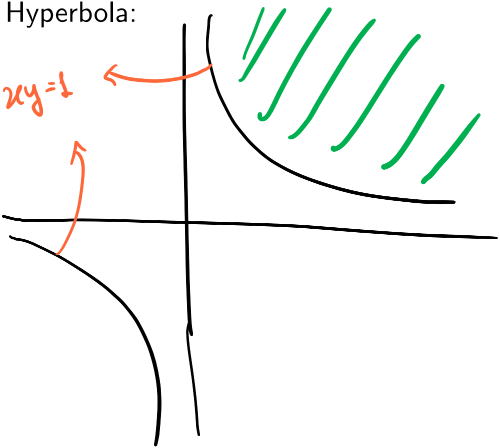
$$1 - x \geq 0$$

$$1 - x^2 - y^2 \geq 0$$



Example of Spectrahedron

Hyperbola:



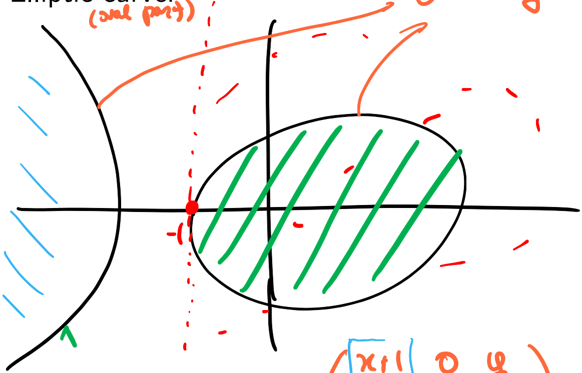
$$\mathcal{H} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x,y \geq 0 \\ xy \geq 1 \end{array} \right\}$$

$$\begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0$$

Example of Spectrahedron

Elliptic curve: (oval part)

$$0 = -2y^2 - x^3 - 3x^2 + x + 3$$



$$\mathcal{E} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} x+1 & 0 & y \\ 0 & 2 & -x-1 \\ y & -x-1 & 2 \end{pmatrix} \succeq 0 \right\}$$

Semidefinite Representations (Spectrahedral Shadows)

For both LPs and SDPs, it is enough to obtain a *linear projection of spectrahedron* (or polyhedron, if in LP).

$$\min \; c^T \bar{x}$$

$$\text{s.t.} \quad A\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \bar{b}, \quad \bar{x}, \bar{y} \geq \bar{0} \qquad P$$

$$\{\bar{x} \mid \exists \bar{y} \text{ s.t. } (x, \bar{y}) \in P\}$$

feasible region

optimizing over projection of a polyhedron!

There are polyhedra with small extension complexity

complex in ambient space

Semidefinite Representations (Spectrahedral Shadows)

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Definition (Projected Spectrahedron/Semidefinite Representation)

A set $S \in \mathbb{R}^n$ is a *projected spectrahedron* if it has the form:

$$S = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \in \mathbb{R}^t \text{ s.t. } \sum_{i=1}^{n} A_i x_i + \sum_{j=1}^{t} B_j y_j \succeq C, \quad A_i, B_j, C \in \mathcal{S}^m \right\}$$

$$T = \{ (x, y) \mid \sum A_i x_i + \sum B_j y_j \succeq C \}$$

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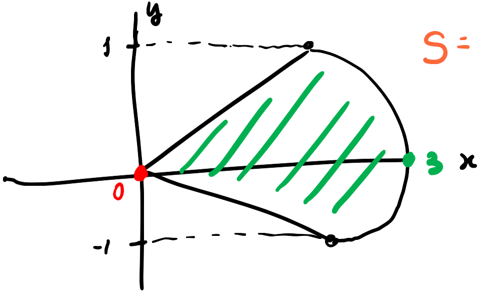
$\min \; c^T x$, $x \in S$ $\iff$ $\min \; c^T x$, $(x, y) \in T$ (SDPs)

Example of Spectrahedral Shadow

Projection quadratic cone intersected with halfspace:



$$S = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \exists z \in \mathbb{R} \text{ s.t. } \begin{pmatrix} z+y & 2z-x \\ 2z-x & z-y \end{pmatrix} \succeq 0,\; z \le 1 \right\}$$

$z \le 1$ $\quad z^2 \geq y^2 + (2z-x)^2$ $\quad \underbrace{z+y \geq 0 \quad z-y \geq 0}_{(z \geq 0)}$

(imply that $x \geq 0$)

know that

$LP \subsetneq SDP \subset HP$

circle

Max-cut

SOS

New object

(hyperbolic programming)

$\min \; c^T x$

s.t. $x \in \Lambda_+(p, \bar{e})$

$\bar{x} \geq 0$

1) is $HP = SDP$?

References I

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