Lecture 21: Hyperbolic Polynomials & Hyperbolicity Cones

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Overview

- Administrivia
- Hyperbolic Polynomials
- Hyperbolicity Cones
- Important Hyperbolic Polynomials
- Conclusion
- Acknowledgements

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Provide us (me and the school) with your evaluation and feedback on the course!

- This would really help me figuring out what worked and what didn't for the course
- And let the school (and santa) know if I was a good boy this term!
- Teaching this course is also a learning experience for me :)

Hyperbolic Polynomials

Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a vector of variables and $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^m$ be a vector in $\mathbb{R}^m$.

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Definition (Hyperbolic Polynomials)

A homogeneous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \ldots, x_m]$ is hyperbolic with respect to a point $\mathbf{e} \in \mathbb{R}^m$ if

- $h(\mathbf{e}) > 0$,
- for each vector $\mathbf{a} \in \mathbb{R}^m$, the univariate polynomial $f(t) := h(t\mathbf{e} - \mathbf{a})$ only has real zeros.

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Example

- $h(\mathbf{x}) = x_1 \cdot x_2 \cdots x_n$, $\mathbf{e} = (1, \ldots, 1)$
- $m = \binom{n+1}{2}$, X symmetric $n \times n$ matrix, $\mathbf{e} = I_n$

$$h(X) = det(X)$$

$h(\bar{x}) = x_1 x_2 \cdots x_n$

$\bar{e} = (1, 1, \ldots, 1)$

1) $h(\bar{e}) = 1$

2) $h(t\bar{e} - \bar{a}) = \underbrace{\prod_{i=1}^{m} (t - a_i)}_{\text{roots are } a_i \in \mathbb{R}}$

$\bar{a} \in \mathbb{R}^m$

$\therefore$ hyperbolic

$$\mathbb{R}^m \cong \{ A \in Mat(n) \mid A \text{ symmetric} \}$$

$$m = \binom{n+1}{2}$$

$$h(X) = \det(X)$$

$$\bar{e} = I_n$$

1) $h(\bar{e}) = \det(I) = 1 > 0$

2) $h(\bar{e}\cdot t - \bar{a}) = \det(t\cdot I - A) = \prod_{i=1}^{n} (t - \lambda_i(A))$

$\updownarrow$

A symmetric

spectral thm $\Rightarrow$ λ_i's are real

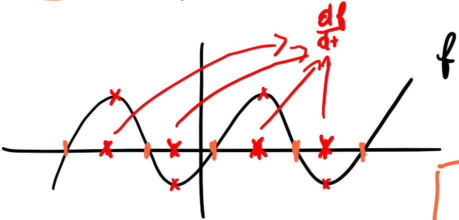
Other Families of Examples

- If a polynomial $h(\mathbf{x})$ is hyperbolic with respect to $\mathbf{e}$, then we know its *directional derivatives* w.r.t. $\mathbf{e}$ are also hyperbolic w.r.t. $\mathbf{e}$

$f(t) = h(t\cdot\bar{e} - \bar{a})$ has all real roots

$\boxed{\frac{d}{dt} f}$ also has all real roots

roots "interlace" roots of f



$\frac{df}{dt} = \partial_{\bar{e}} h(t\bar{e} - \bar{a})$

$\boxed{g(x) = \partial_{\bar{e}} h(\bar{x})}$

Other Families of Examples

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- With this observation, we get that the *elementary symmetric polynomials* are also hyperbolic w.r.t. $\mathbf{e} = (1, 1, \dots, 1)$.

$$h(\bar{x}) = x_1 x_2 \cdots x_m \qquad \bar{e} = (1, 1, \dots, 1)$$

$$\partial_{\bar{e}} h(\bar{x}) = x_2 \cdots x_m + x_1 x_3 \cdots x_m + x_1 x_2 x_4 \cdots x_m + \cdots$$

$$= E_{m,m-1}(\bar{x})$$

$$\partial_{\bar{e}}^{k} h = E_{m,m-k}(\bar{x})$$

elementary symmetric polynomials.

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- The product of two hyperbolic polynomials w.r.t. $\mathbf{e}$ is also hyperbolic w.r.t. $\mathbf{e}$

$g(\bar{x})$, $h(\bar{x})$ $\bar{e}$

$\underbrace{g(t\bar{e}-\bar{a})}\,\underbrace{h(t\bar{e}-\bar{a})}$ real rooted

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- The sum of two hyperbolic polynomials w.r.t. $\mathbf{e}$ is *not necessarily* hyperbolic!

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- With this observation, we get that the *elementary symmetric polynomials* are also hyperbolic w.r.t. $\mathbf{e} = (1, 1, \ldots, 1)$.
- The product of two hyperbolic polynomials w.r.t. $\mathbf{e}$ is also hyperbolic w.r.t. $\mathbf{e}$
- The sum of two hyperbolic polynomials w.r.t. $\mathbf{e}$ is *not necessarily* hyperbolic!
- The Lorenz polynomial

$$h(x_0, x_1, \ldots, x_n) := x_0^2 - x_1^2 - \cdots - x_n^2$$

is hyperbolic w.r.t. $\mathbf{e} = (1, 0, \ldots, 0)$

$$h(t\bar{e} - \bar{a}) = t^2 - \underbrace{\sum_{i=1}^{n} a_i^2}_{\geq 0}$$

Eigenvalues

- Given a hyperbolic polynomial $h(\mathbf{x})$ w.r.t. $\mathbf{e}$ and another point $\mathbf{a}$, we say that the roots of
$$p(\lambda \mathbf{e} - \mathbf{a})$$
are the *eigenvalues* of $\mathbf{a}$

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- This generalizes the definition of eigenvalues of symmetric matrices, where in this case the hyperbolic polynomial is $\det(X)$ and $\mathbf{e} = I_n$.

roots of $\det(tI - A)$

Eigenvalues

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- This generalizes the definition of eigenvalues of symmetric matrices, where in this case the hyperbolic polynomial is $\det(X)$ and $\mathbf{e} = I_n$.
- If $h(\mathbf{x})$ has degree d, let's write the eigenvalues of $\mathbf{a}$ as

$$\lambda_1(\mathbf{a}) \leq \lambda_2(\mathbf{a}) \leq \cdots \leq \lambda_d(\mathbf{a})$$

  Some simple properties of the eigenvalues:

  1. $\lambda_j(s\mathbf{a} + t\mathbf{e}) = \begin{cases} s\lambda_j(\mathbf{a}) + t, & \text{if } s \geq 0 \\ s\lambda_{n-j}(\mathbf{a}) + t, & \text{if } s < 0 \end{cases}$
  2. $h(\mathbf{a}) = h(\mathbf{e}) \cdot \prod_j \lambda_j(\mathbf{a})$

$h(t\bar{e} - \bar{a}) = h(\bar{e}) \cdot \prod_{j=1}^{d} (t e_j - \lambda_j(a))$

$(-1)^d \cdot h(\bar{a}) = h(-\bar{a}) = h(\bar{e}) \cdot (-1)^d \cdot \prod_j \lambda_j(a)$

Hyperbolicity Cones

Definition (Hyperbolicity Cones)

Given $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_m]$ hyperbolic with respect to $\mathbf{e} \in \mathbb{R}^m$, define its hyperbolicity cone as the set

$$\Lambda_+(h, \mathbf{e}) = \{\mathbf{a} \in \mathbb{R}^m \mid \underbrace{\text{all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are non-negative}}_{\text{eigenvalues of } \bar{a} \text{ non-negative}}\}$$

cone: $\bar{a} \in \Lambda_+$ then $\alpha \cdot \bar{a} \in \Lambda_+$ for any $\alpha > 0$.

$\bar{e} \in \Lambda_+(h, \bar{e})$ $\quad h(t\bar{e} - \bar{e}) = h((t-1)\cdot\bar{e})$

$= (t-1)^d \cdot h(\bar{e})$

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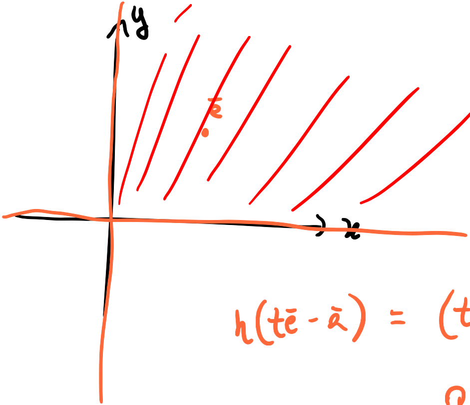
Theorem ([Gårding, 1959])

- $\Lambda_+(h, \mathbf{e})$ *is a closed convex cone*
- *Equivalent definition of* $\Lambda_+(h, \mathbf{e})$*: closure of connected component of* $\{\mathbf{a} \in \mathbb{R}^m \mid h(\mathbf{a}) \neq 0\}$ *that contains* $\mathbf{e}$.

$h(\bar{x}) = x_1 x_2 \cdots x_m$ $\bar{e} = (1, 1, \dots, 1)$

$m = 2$ $x_1 = x$ $x_2 = y$

$h(x, y) = xy$



connected component that contains $\bar{e}$

$h(t\bar{e} - \bar{a}) = (t - a_1)(t - a_2)$

$a_i \geq 0$

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Theorem ([Gårding, 1959])

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- Origins in PDE in works of Petrovsky and Gårding.
- Convex structure can be used for optimization [Güler, 1997]!
- Recent applications in combinatorics and optimization [Gurvits, 2004].

$h(X) = \det(X)$

$\Lambda_+(\det(x), I) =$ PSD matrices

$\Lambda_{++}(\det, I) =$ Positive definite matrices

Hyperbolicity Cones are Connected

- We will prove that $\Lambda_+(h, \mathbf{e})$ is the closure of connected component of $\{\mathbf{a} \in \mathbb{R}^m \mid h(\mathbf{a}) \neq 0\}$ that contains $\mathbf{e}$. Let's define:

$$\Lambda_{++}(h, \mathbf{e}) = \{\mathbf{a} \in \mathbb{R}^m \mid \text{ all roots of } h(t\mathbf{e} - \mathbf{a}) \text{ are positive}\}$$

by continuity of eigenvalues Λ_{++} is the interior of Λ_+.

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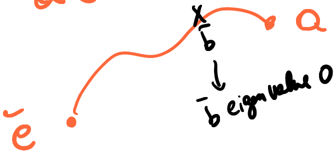
- Let S be the connected component of $h(\mathbf{x}) \neq 0$ which contains $\mathbf{e}$.

By continuity of eigenvalues, $\boxed{S \subset \Lambda_{++}}$

$\bar{a} \in S$



$\Rightarrow \bar{a} \in \Lambda_{++}$

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  1. Let $\mathbf{a} \in \Lambda_{++}$ and let ℓ be the line segment with endpoints $\mathbf{a}$ and $\mathbf{e}$

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  2. By compactness of the line segment and $\mathbf{a}, \mathbf{e} \in \Lambda_{++}$, there is $\alpha \in \mathbb{R}$ such that for all $\mathbf{y} \in \ell$

$$\underbrace{h(t\mathbf{e} - \mathbf{y})}_{f(\bar{y})} > 0 \text{ when } t \geq \alpha \quad \text{large enough}$$

$f : \underset{\sim}{\ell} \to \mathbb{R}$

$h(\bar{e}) > 0$

$h(\bar{a}) = h(\bar{e}) \prod_{j=1}^{d} \lambda_j(\bar{a}) > 0$

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     $a + \alpha\bar{e}$

     $\lambda_j(\bar{a} + t\bar{e}) = \lambda_j(\bar{a}) + t > 0$
  3. Now consider the path $P_1 \cup P_2 \cup P_3$ where

$h(\bar{a} + t\bar{e}) > 0 \leftarrow$ $$P_1 = \{\mathbf{a} + t\mathbf{e} \mid 0 \leq t \leq \alpha\}$$

$h(\bar{y} + \alpha\bar{e}) \leftarrow$ $$P_2 = \{(1-\beta)\mathbf{a} + \beta\mathbf{e} + \alpha\mathbf{e} \mid 0 \leq \beta \leq 1\}$$

$\bar{y} \in \ell$ $\quad h(\bar{y} + t\alpha\bar{e}) > 0$

$$P_3 = \{\mathbf{e} + t\mathbf{e} \mid 0 \leq t \leq \alpha\}$$

$h(\bar{e} + t\bar{e}) > 0$

$= (t+1)^d \cdot h(\bar{e})$

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  4. Since h remains strictly positive through this path, it is still in S

Hyperbolicity in Λ_{++}[1]

- We'll prove: if $\mathbf{b} \in \Lambda_{++}(h, \mathbf{e})$, then h is also hyperbolic w.r.t. $\mathbf{b}$.

$$\Lambda_{++}(h, \mathbf{e}) = \Lambda_{++}(h, \mathbf{b}) \qquad \text{by connected component definition}$$

[1] Shortest path between two truths in the real domain passes through the complex domain. - Hadamard

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- $\mathbf{b} \in \Lambda_{++} \Rightarrow h(\mathbf{b}) > 0$

$$h(\bar{b}) = h(\bar{e}) \prod_{j=1}^{m} \lambda_j(\bar{b}) > 0$$

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- $\mathbf{b} \in \Lambda_{++} \Rightarrow h(\mathbf{b}) > 0$ WTS: $h(t\bar{b} - \bar{a})$ real rooted
- Let $\mathbf{a} \in \mathbb{R}^m$ and $\alpha > 0$

For all $s \geq 0$ the roots of $h_s(t) := h(\alpha i\mathbf{e} + s\mathbf{a} - t\mathbf{b})$ have negative imaginary part.

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  1. True if $s = 0$, since $\mathbf{b} \in \Lambda_{++}$

$$h(\alpha i \mathbf{e} - t\mathbf{b}) = h(\mathbf{e}) \cdot \prod_j (\alpha i - t\lambda_j(\mathbf{b}))$$

$$h(t\bar{e} - \bar{b}) = h(\bar{e}) \cdot \prod_{j=1}^{t} (t - \lambda_j(\bar{b}))$$

$$\mu_j(b) = \frac{-\alpha i}{\lambda_j(b)}$$

purely negative imaginary

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  $$h(\alpha i \mathbf{e} - t\mathbf{b}) = h(\mathbf{e}) \cdot \prod_j (\alpha i - t\lambda_j(\mathbf{b}))$$

  2. If for some $s > 0$ a root of $h_s(t)$ has non-negative imaginary part, by continuity of roots w.r.t. s, we would have $0 < s' \leq s$ such that $h_{s'}(t)$ has a real root t^*.

$h_0(t)$ negative imaginary roots $\longrightarrow$ $h_1(t)$ non-negative imaginary part

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  $\Lambda_{++}(h, \mathbf{e}) = \Lambda_{++}(h, \mathbf{b})$ by connected component definition

- $\mathbf{b} \in \Lambda_{++} \Rightarrow h(\mathbf{b}) > 0$
- Let $\mathbf{a} \in \mathbb{R}^m$ and $\alpha > 0$

  For all $s \geq 0$ the roots of $h_s(t) := h(\alpha i\mathbf{e} + s\mathbf{a} - t\mathbf{b})$ have negative imaginary part.

  1. True if $s = 0$, since $\mathbf{b} \in \Lambda_{++}$

     $$\boxed{h(\alpha i \mathbf{e} - t\mathbf{b}) = h(\mathbf{e}) \cdot \prod_j (\alpha i - t\lambda_j(\mathbf{b}))}$$

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  3. Thus, s' is a root of $h(\alpha i\mathbf{e} + s'\mathbf{a} - t^*\mathbf{b})$, which contradicts hyperbolicity of h w.r.t. $\mathbf{e}$.

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$$h\left(\alpha_i \bar{e} + s'\bar{a} - t^*\bar{b}\right)$$

$$= h(\bar{e}) \cdot \prod_{j=1}^{d-1} \left(t - \cdots\right) \cdot \left(t - t^*\right)$$

$h(z\bar{e} - \bar{c})$ only has real roots

$$\boxed{h\left(z\bar{e} + s'\bar{a} - t^*\bar{b}\right)}$$

$$\left(\frac{z}{\alpha_i} - t^*\right) \text{ root}$$

$$h\left(z\bar{e} - \underbrace{\left(t^*\bar{b} - s'\bar{a}\right)}_{t^*,\, s' \in \mathbb{R}}\right)$$ <u>not</u> <u>real</u> <u>rooted</u>

$t^*\bar{b} - s'\bar{a} \in \mathbb{R}^m$

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- Let $\mathbf{a} \in \mathbb{R}^m$ and $\alpha > 0$

  For all $s \geq 0$ the roots of $h_s(t) := h(\alpha i \mathbf{e} - s\mathbf{a} - t\mathbf{b})$ have negative imaginary part.

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- Let $\mathbf{a} \in \mathbb{R}^m$ and $\alpha > 0$

  For all $s \geq 0$ the roots of $h_s(t) := h(\alpha i\mathbf{e} + s\mathbf{a} - t\mathbf{b})$ have negative imaginary part.
- If we let $\alpha \to 0$, continuity of roots w.r.t. α implies that *all roots* of $h(t) := h(s\mathbf{a} - t\mathbf{b})$ have non-positive imaginary part.

$$\lim_{\alpha \to 0} \underbrace{h(\alpha i\bar{e} + s\bar{a} - t\bar{b})}_{\text{negative}} = \underbrace{h(s\bar{a} - t\bar{b})}_{\text{non-positive}}$$

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- $h(t) := h(s\mathbf{a} - t\mathbf{b})$ is a real univariate polynomial, so complex roots have to appear in complex conjugate pairs. Thus all roots must be real.

Since $\mathbf{a}$ was arbitrary, this proves ~~$\mathbf{b} \in \Lambda_{++}$~~.

h hyperbolic w.r.t. $\underline{}$ b

Hyperbolicity Cones are Convex

- Take $\mathbf{a}, \mathbf{b} \in \Lambda_{++}(h, \mathbf{e})$ and $\alpha, \beta > 0$. We want to show that

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- But the eigenvalues of

$\bar{b} = \bar{e}$

$$h(t\mathbf{e} - (\alpha \mathbf{a} + \beta \mathbf{b}))$$

are exactly

$$\alpha \lambda_j(\mathbf{a}) + \beta > 0$$

$\alpha > 0, \beta > 0 \quad \bar{a} \in \Lambda_{++}$

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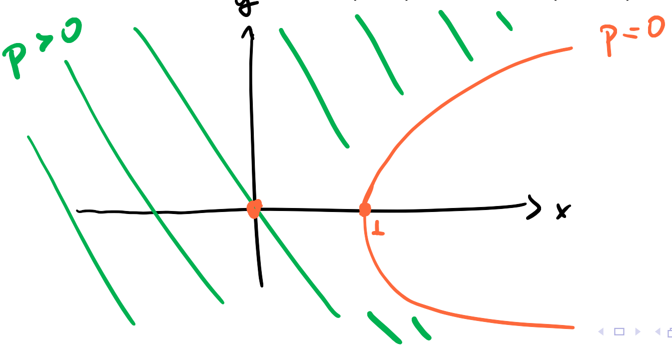

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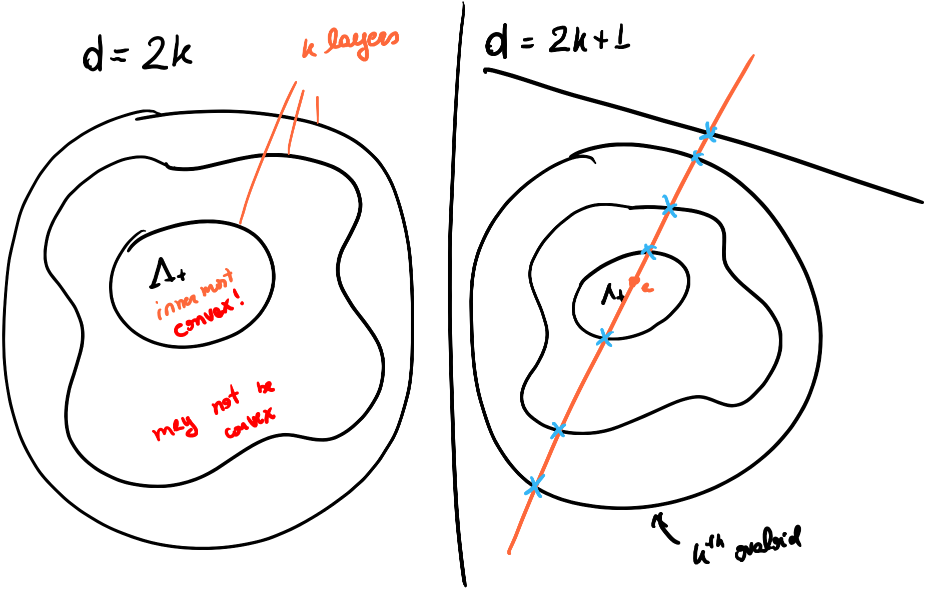
How do the zeros of hyperbolic polynomials look?



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- If $\mathcal{C}$ is an algebraic interior, then a minimal degree polynomial defining $\mathcal{C}$ is *unique* (up to units) *minimal defining polynomial* of $\mathcal{C}$
- If p is the minimal defining polynomial of $\mathcal{C}$, any other defining polynomial q of $\mathcal{C}$ must be a *multiple* of p in the following way:

  $$q(\mathbf{x}) = p(\mathbf{x}) \cdot h(\mathbf{x})$$

  where h is *strictly positive* on a *dense connected subset* of $\mathcal{C}$

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- In particular, we know that hyperbolicity cones are semialgebraic, since

$$\Lambda_+(h, \mathbf{e}) := \{\mathbf{a} \in \mathbb{R}^m \mid E_{m,d}(\lambda_1(\mathbf{a}), \ldots, \lambda_{\deg(h)}(\mathbf{a})) \geq 0 \ \forall d \in [\deg(h)]\}$$

$\lambda_1(\bar{a}), \lambda_2(\bar{a}), \ldots, \lambda_d(\bar{a}) \geq 0 \Leftarrow\Rightarrow \underbrace{E(\lambda_j\text{'s}) \geq 0}_{\text{polynomials in } \bar{a}}$

- Administrivia
- Hyperbolic Polynomials
- Hyperbolicity Cones
- Important Hyperbolic Polynomials
- Conclusion
- Acknowledgements

Real-Stable Polynomials

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     where A_i's are PSD and B is Hermitian.
- A polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is real stable iff for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$ with $\mathbf{b} > 0$, the univariate polynomial $p(\mathbf{a} + \mathbf{b}t)$ only has real roots

Real Stability and Hyperbolicity

- Let $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ be a homogeneous polynomial. Then

$$p(\mathbf{x}) \text{ is real-stable iff } p(\mathbf{x}) \text{ is hyperbolic w.r.t. } \mathbf{e} = (1, \ldots, 1) \text{ and } \mathbb{R}^m_+ \subset \Lambda_{++}(p, \mathbf{e})$$

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- Corollary of the lemma in previous slide.
- Up to a linear change of coordinates every hyperbolic polynomial can be made real stable
- Real-stable polynomials have many nice properties, so approaching hyperbolic questions via real stability can be quite useful

Matching Polynomials

- Given a graph $G(V,E)$ the (homogeneous multivariate) *matching polynomial*

$$\mu_G(\mathbf{x},\mathbf{w}) = \sum_{M\in\mathcal{M}(G)} (-1)^{|M|} \prod_{u\notin V(M)} x_u \cdot \prod_{e\in M} w_e^2$$

is hyperbolic w.r.t. $\mathbf{e} = (1_V, \mathbf{0})$.

Conclusion

- Hyperbolic polynomials
- Hyperbolicity cones are convex algebraic interiors
- Relationship to real-stability

Acknowledgement

- Lecture based largely on [Renegar 2004] and [Vinnikov 2012]

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