Lecture 20: Introduction to Convex Algebraic Geometry II

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Overview

- Basic Definitions and Isolation Theorem
- Faces and Extreme Points
- Conclusion

Affine Subspaces, Hulls & Linear Functionals

- Let V be a $\mathbb{R}$-vector space and $L \subset V$ be a subspace
- Any translation $A = L + u$ is called *affine subspace* of V

$a, b \in A$

$\lambda a + (1-\lambda) b \in A \quad \lambda \in \mathbb{R}$

$a = a' + u$

$b = b' + u$

$c = c' + u$

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for all $\alpha, \beta \in \mathbb{R}$, $u, v \in V$.

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  for (non-zero) linear functional f and $\alpha \in \mathbb{R}$
- A linear combination

$$\alpha_1 v_1 + \cdots + \alpha_n v_n, \text{ where } \alpha_1 + \cdots + \alpha_n = 1$$

  is an *affine combination*. The *affine hull* of a set $X \subset V$, denoted $\mathit{aff}(X)$, is the set of all affine combinations of points of X.

Quotients, Projections and Codimension

- Let V be a $\mathbb{R}$-vector space and $L \subset V$ be a subspace
- The *quotient space* V/L is the set of affine subspaces parallel to L, i.e.

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- The linear transformation

$$\Pi : V \to V/L, \quad \Pi(u) = L + u$$

is the *projection* onto the quotient space V/L

Halfspaces, Isolation and Separation

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- The complement

$$V \setminus H = H_+ \sqcup H_-$$

union of two open convex sets, the *open halfspaces*

$$H_+ = \{u \in V \mid f(u) > \alpha\}$$

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- A hyperplane $H \subset V$ *isolates* a set $A \subset V$ iff $A \subset \overline{H_-}$ or $A \subset \overline{H_+}$.
  H *strictly isolates* A iff $A \subset H_-$ or $A \subset H_+$.

A is in the closure of one of the half-spaces

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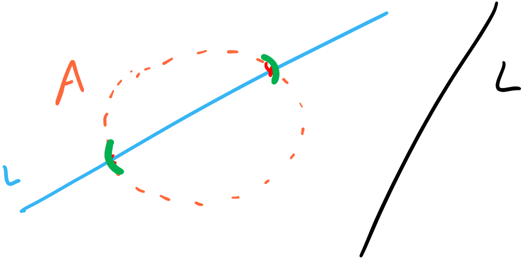
- A hyperplane $H \subset V$ *separates* sets $A, B \subset V$ iff $A \subset \overline{H_-}$ AND $B \subset \overline{H_+}$ (or vice-versa).
  H *strictly separates* A and B iff $A \subset H_-$ AND $B \subset H_+$ (or vice-versa).

Algebraically Open Subsets

- Let V be $\mathbb{R}$-vector space and $A \subset V$
- A is *algebraically open* if the intersection of A with every straight line in V is an *open interval* (possibly empty)



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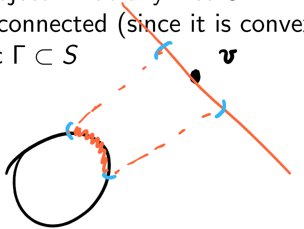

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$$\Gamma = \{(\cos\theta, \sin\theta) \mid \alpha < \theta < \beta\}$$

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  5. $\alpha - \beta \leq \pi$, otherwise Γ would contain antipodal points and thus $0 \in A$

$x_1 - x$

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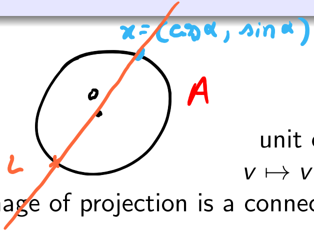

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5. $\alpha - \beta \leq \pi$, otherwise Γ would contain antipodal points and thus $0 \in A$
6. Thus, if $y \in \mathbb{R}^2$ is an endpoint of Γ (thus not in Γ), the line through 0 and y is our desired hyperplane.

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- Take any plane $B \subset V$ such that $0 \in B$.
  - $B \cap A$ is a convex algebraically open set.
  - By previous slide, we have a line $L \subset B$ through 0 such that $L \cap A = \emptyset$.

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- We'll prove H is a hyperplane:
  1. Let $\Pi : V \to V/H$ be the projection map
  2. If H not hyperplane, $\dim(V/H) \geq 2$ and $\Pi(A)$ convex algebraically open subset of V/H

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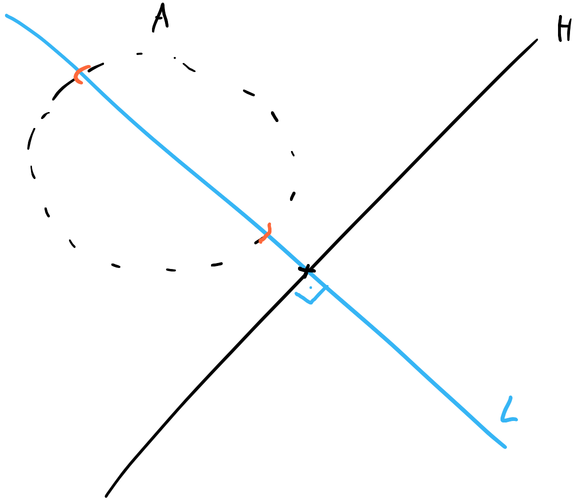
  3. By previous slide, there is a line $L \in V/H$ such that $0 \in L$ and $L \cap \Pi(A) = \emptyset$
  4. Taking preimage of L we contradict maximality of H

$H + L \supset H \quad (H+L) \cap A = \emptyset$


A
H
L

Interior and Boundary

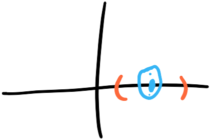
- $A \subset \mathbb{R}^d$ be a set
- Point $u \in A$ is an *interior point* of A if there is $\varepsilon > 0$ such that

$$B_\varepsilon(u) \subset A$$

the *interior* of A

$$int(A) := \{u \in A \mid u \text{ interior point of } A\}$$

$$B_\varepsilon(u) = \{ v \in \mathbb{R}^d \mid \|u - v\|_2 < \varepsilon \}$$



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Theorem (Empty interior implies not full dimensional)

Let $A \subset \mathbb{R}^d$ be convex. If $int(A) = \emptyset$ then there is an affine subspace $L \subset \mathbb{R}^d$ such that $A \subset L$ and $\dim(L) < d$.

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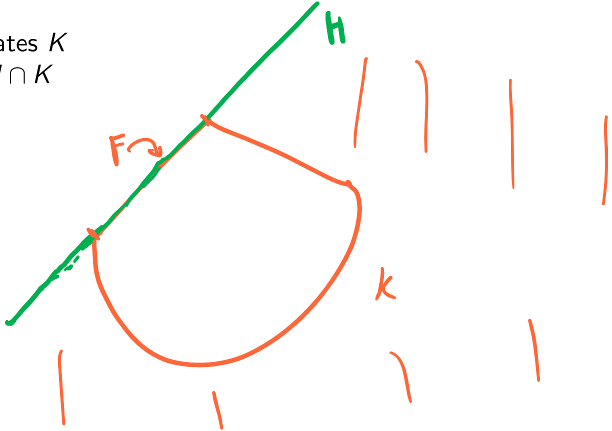
- The *dimension* of a convex set A is the dimension of the *smallest* affine subspace that contains A. Convention: $\dim(\emptyset) = -1$.

Faces and Exposed Points

- $K \subset \mathbb{R}^d$ be a closed convex set
- A (possibly empty) set $F \subset K$ is a *face* of K if there is a hyperplane H which
  1. H isolates K
  2. $F = H \cap K$



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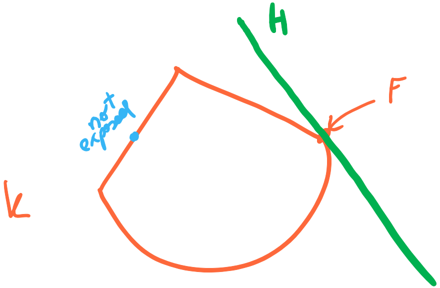

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- If F is a point, then it is called an *exposed point* of K
- If $F \neq \emptyset, K$ then F is a *proper face* of K
- The following theorem relates the boundary and faces

Theorem (Boundary point lies in a face)

Let $K \subset \mathbb{R}^d$ be a convex set where $int(K) \neq \emptyset$. Let $u \in \partial K$ be a point. Then, there exists a hyperplane H - called a support hyperplane at u such that $u \in H$ and H isolates K.

Extreme Points

- V be a $\mathbb{R}$-vector space and $A \subset V$ be a set.
- A point $u \in A$ is an *extreme point* of A if for any two points $b, c \in A$ such that $u = \dfrac{b+c}{2}$, we have $u = b = c$

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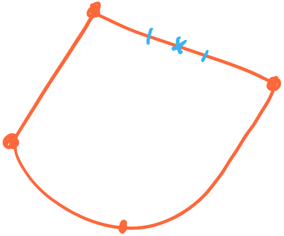

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  2. If f attains its *maximum* (or *minimum*) α on A, let

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     then $ex(B) \subset ex(A)$.

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Theorem (Minkowski's Theorem - or finite-dimensional Krein-Milman)

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- if $u \in \partial K$ then there is a face $F \subset K$ that contains u and we are done by induction, since $ex(F) \subset ex(K)$

Minkowski's Theorem

Theorem (Minkowski's Theorem - or finite-dimensional Krein-Milman)

Let $K \subset \mathbb{R}^d$ be a compact convex set. Then K is the convex hull of the set of its extreme points.

$$K = conv(ex(K))$$

- Proof by induction on dimension d.
- Base case: $d = 0$. In this case K is a point and the result follows.
- If $d > 0$, we can assume that $int(K) \neq \emptyset$, otherwise K lies in an affine subspace of smaller dimension
- Now let's show that each point $u \in K$ can be written as a convex combination of $ex(K)$
- if $u \in \partial K$ then there is a face $F \subset K$ that contains u and we are done by induction, since $ex(F) \subset ex(K)$
- $u \in int(K)$ and $u \in L$ a line, then compact convex $K \Rightarrow L \cap K$ is an interval $L \cap K = [a, b]$, where $a, b \in \partial K$.

Extreme points and Optimization

- If $K \subset \mathbb{R}^d$ is compact and convex, and $f : \mathbb{R}^d \to \mathbb{R}$ be a linear functional, then there is $u \in ex(K)$ such that $f(u) \geq f(x)$ for all $x \in K$.

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- Let $F = \{u \in K \mid f(u) = \alpha\}$
- F is also compact convex, and non-empty
- Minkowski's theorem implies $ex(F) \neq \emptyset$
- $ex(F) \subset ex(K)$ finishes the proof.

$u \in ex(F) \subset ex(K)$

$f(u) = \alpha$

Conclusion

- Today we learned more basic definitions on convex sets
- Isolation theorem
- Basics on faces and extreme points
- Minkowski's Theorem
- Importance of extreme points in optimization

References I

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