Lecture 2: Algebraic Circuits & Algebraic Complexity

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

- Algebraic Complexity Classes
- Structural Results on Algebraic Circuits
- Conclusion
- Acknowledgements

Complexity Measures in Algebraic Circuits

- *circuit size:* number of edges in the circuit, denoted by $\mathcal{S}(\Phi)$
- *cost of ring elements:* in classical algebraic complexity, there is unit cost for the use of any base ring element
- Sometimes we will add bit complexity of base ring elements

$P(x,y) = x^2 + 2xy - 3y + 1$

sparse reprentation

$(1, x^2)$, $(2, xy)$, $(-3, y)$, $(1, 1)$

dense representation

$[2, (1, x^2), (2, xy), (0, y^2), (0, x), (-3, y), (1, 1)]$

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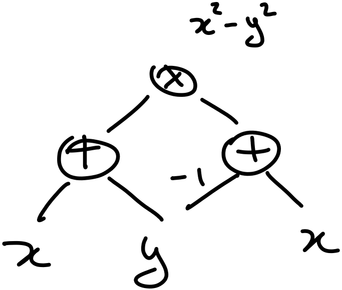
- *circuit depth:* length of longest direct path from an input to an output



depth-2

depth = parallel complexity

Complexity Measures in Algebraic Circuits

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- Sometimes we will add bit complexity of base ring elements
- *circuit depth:* length of longest direct path from an input to an output
- *constant depth circuits:* for circuits of constant depth, we don't place restriction on the fan-in of an edge.

Convention: whenever we don't specify depth, the fanin of each gate is ≤ 2

circuits of constant depth may have arbitrary fanin.

Examples - Constant Depth Circuits

$\Sigma\Pi$ - ckts $\longleftrightarrow$ sparse polynomials

$\prod_{i=1}^{n}(x_i+1)$ $\quad 2^n$ size $\Sigma\Pi$ - ckts

$\Sigma\Pi\Sigma$ -ckts $\longleftrightarrow$ $\sum_{i=1}^{s}\prod_{j=1}^{d} \ell_{ij}(x_1,\dots,x_n)$

~~$\Pi\Sigma\Pi$ prod. of sparse poly~~

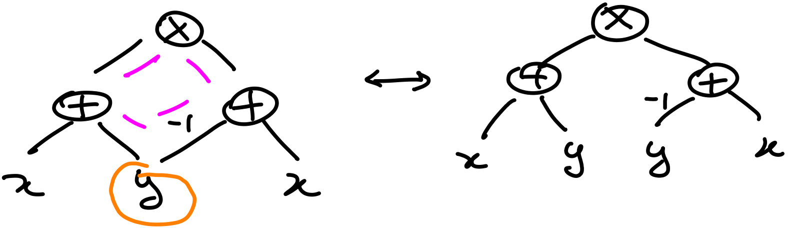
$\prod_{i=1}^{n}(x_i+1)$ $\quad O(n)$ size $\Sigma\Pi\Sigma$ - ckt

$\Sigma\Pi\underbrace{\Sigma\Pi}$ - ckts $\longleftrightarrow$ $\sum_{i=1}^{s}\prod_{j=1}^{d} \overbrace{P_{ij}}^{\Sigma\Pi}$

P_{ij} sparse polynomials

Algebraic Formulas

- when the computation graph is a tree (i.e., we don't reuse computation) we get an algebraic formula



Result: P has ckt of size s, then

P has a formula of size $s^{\log d \log n} = s^{\log^2 n}$

poly-size ckt $\subset$ quasi-poly formula

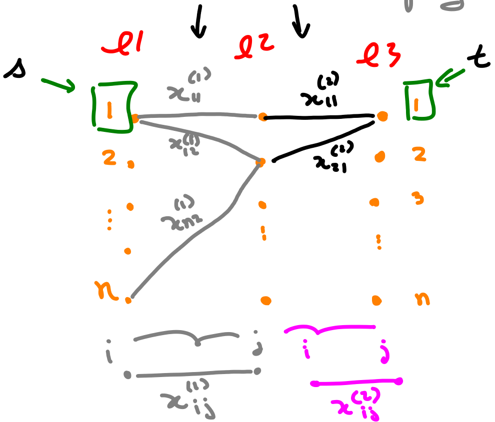
Algebraic Branching Programs

- polynomials which are projections of the *Iterated Matrix Multiplication* (IMM) polynomial

$$X_i = \left(x^{(i)}_{jk}\right)^{n}_{j,k=1}$$

$$\underbrace{\mathrm{tr}[X_1 X_2 \cdots X_d]}_{\text{poly in } n^2 \cdot d \text{ variables}}$$

d degree

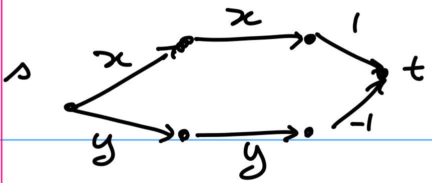


$$P_{1,1} = \sum_k x^{(1)}_{1k} \cdot x^{(2)}_{k1}$$

$$[X_1 X_2]_{11} = P_{11}$$

$$P = X_1 X_2$$

$$P_{ij}$$



expressed as $t(X_1 X_2 X_3)$

$$\sum_{s-t \text{ path}} \prod (\text{vars in path})$$

$$x \cdot x \cdot 1 + y \cdot y \cdot (-1) = x^2 - y^2$$

Algebraic branching program

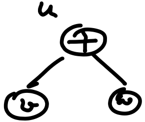
Homogeneous Components

Theorem ([Strassen 1973])

If a polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ can be computed by a circuit Φ of size $\mathcal{S}(\Phi)$, then the homogeneous components $H_0[p], H_1[p], \ldots, H_r[p]$ can be computed by a homogeneous circuit of size $O(r^2 \cdot \mathcal{S}(\Phi))$.

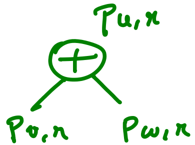
Proof: induction over depth.

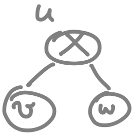
Base: input gates homogeneous.



$p_u = p_v + p_w$

$H_r[p_u] = H_r[p_v] + H_r[p_w]$

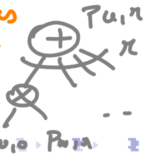




$p_u = p_v \cdot p_w$

$$H_r[p_u] = \sum_{d=0}^{r} \underbrace{H_d[p_v]}_{d} \cdot \underbrace{H_{r-d}[p_w]}_{r-d}$$

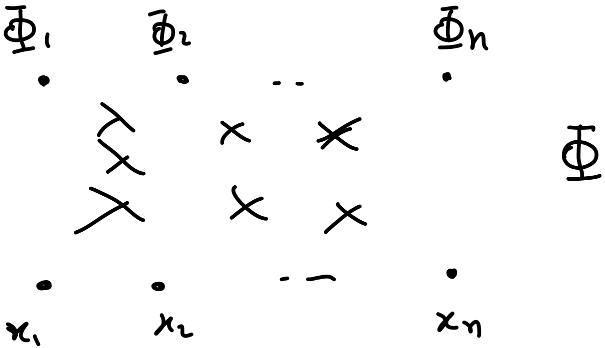
$O(\log r)$ r gates ←



Universal Circuits

Definition

A circuit Φ is called (n, d, s)-*universal*, if the following holds:
If $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ are homogeneous polynomials of degree d which can be simultaneously computed by a circuit of size s, then there is a circuit $\boxed{\Psi}$ computing $f_1, \ldots, f_n$ with same computation graph as $\boxed{\Phi}$.



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- Φ is (n, d, s)-universal if any circuit Ψ of size $\leq s$ computing homogeneous polynomials of degree d are a projection of Φ

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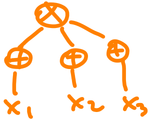
- Φ is (n, d, s)-universal if any circuit Ψ of size $\leq s$ computing homogeneous polynomials of degree d are a projection of Φ
- Normal-homogeneous form:
  - all input gates are labelled by a variable
  - all edges leaving input gates are connected to sum gates
  - all output gates are sum gates
  - alternating sum-product layers
  - fanin of each *product* gate is exactly 2
  - out-degree of each *addition* gate is ≤ 1



formula like for addition

$\Sigma \, \Pi \, \Sigma \, \Pi \, \Sigma \, \Pi \, \Sigma$

Universal Circuits

Theorem ([Raz 2008])

For any integers $s \geq n$ and d, we can construct in time $\mathrm{poly}(s, d)$ a circuit Φ in normal-homogeneous form with at most $O(s^4 d)$ nodes that is (n, d, s)-universal.

$s \cdot d^4$

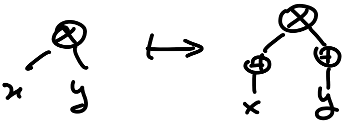
- For every circuit Ψ, there is a circuit χ in normal homogeneous form computing all polynomials that Ψ computes

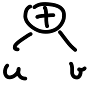
$O(s \cdot d^2)$

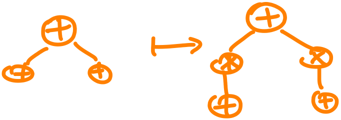
s

$\Psi \longmapsto \underbrace{H_d[\Psi]}_{\text{homogeneous}}$

P $\qquad$ P







Universal Circuits

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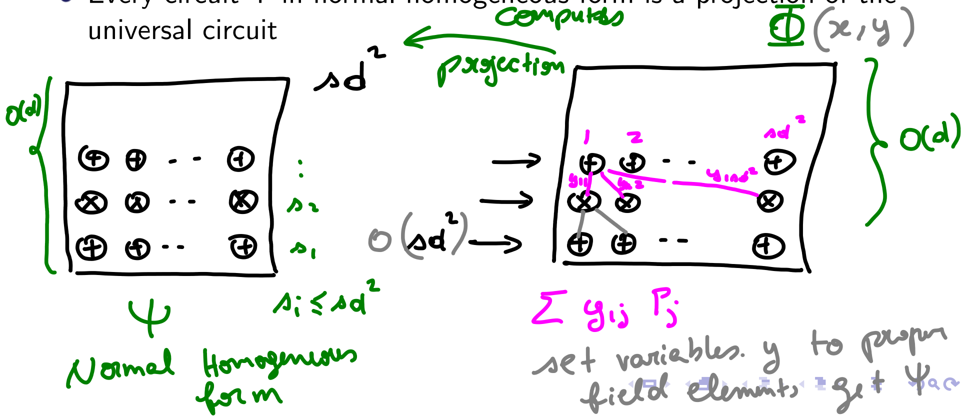
- Every circuit Ψ in normal homogeneous form is a projection of the universal circuit



Almost all polynomials are hard to compute

Corollary: the set of polynomials which can be computed by small circuits has "measure zero" over $\mathbb{F}[x_1, .., x_n]$.

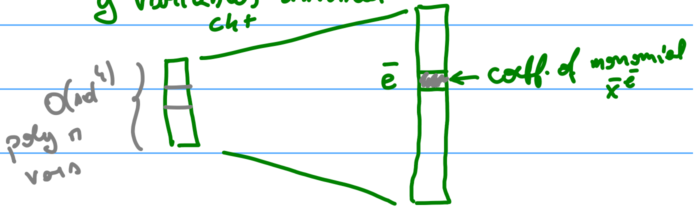
Proof: look $\mathbb{F}[x_1, .., x_n]_d$

dim $\binom{n+d-1}{d}$ d^n or n^d

if polynomial $f \in \mathbb{F}[x_1, .., x_n]_d$ computed by ckt of size s then f can be computed by projection of universal ckt $O(sd^4)$ $(\mathrm{poly}(n))$

$$\Phi : \underbrace{\mathbb{F}^{O(sd^4)}}_{y \text{ variables universal ckt}} \longrightarrow \mathbb{F}[x_1,\dots,x_n]_d$$

$g_{\bar{e}}(y_1,\dots,y_{O(sd^4)})$



universal ckt

$$\mathrm{Im}(\Phi) \supset \left\{ f \;\middle|\; \begin{array}{c} f \text{ computed} \\ \text{by small} \\ \text{ckts} \end{array} \right\}$$

$$\dim\left(\overline{\mathrm{Im}(\Phi)}\right) \le O(sd^4)$$

$$\ll \binom{n+d-1}{d}$$

$\Rightarrow$ most polys hard.

Computing Partial Derivatives

$\Phi^{w} \to \Phi^{w}_{y}$

Theorem ([Baur, Strassen 1983])

If a polynomial $p(x_1, \dots, x_n) \in \mathbb{F}[x_1, \dots, x_n]$ can be computed by a circuit Φ of size s and depth d, then there is a circuit Ψ of size $O(s)$ and depth $O(d)$ computing (simultaneously) the polynomials $\partial_1 p, \partial_2 p, \dots, \partial_n p$.

Notation: v, w two gates in Φ

computing $f_v, f_w \in \mathbb{F}[x_1, \dots, x_n]$

if delete wires into w (make it "input") and label it by new variable y, then

- $\Phi_{w=y}$ is our new ckt
- $f_{v,w}(x_1, \dots, x_n, y)$ polynomial computed in gate v in $\Phi_{w=y}$.

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- Taking partial derivative with respect to a gate

$$\underbrace{\partial_w f_v}_{\in \mathbb{F}[x_1, \ldots, x_n]} := \left(\partial_y \underbrace{f_{v,w}}_{} \right)\Big|_{\underline{y = f_w}}$$

partial derivative w.r.t. w of gate v

w input gate with variable y

substitute y by f_w

Computing Partial Derivatives

Φ original

Ψ partial derivatives

- Induction on circuit size

Base case: input gates ✓

Induction ckt size: Φ let w be the lowest gate that we have not computed $\Psi_{\partial_1 f_w} \dots \Psi_{\partial_n f_w}$

$S(\Phi_{w=y}) \leq S(\Phi) - 1$ (removed two edges)

By induction can compute $\Psi_{w=y}$

Computing Partial Derivatives

Chain rule:

$$\partial_i f_v = \underbrace{\partial_i f_{v,w} \Big|_{y=f_w}}_{\text{in } \Psi_{w=y}} + \underbrace{\partial_y f_{v,w} \Big|_{y=f_w}}_{\text{in } \Psi_{w=y}} \cdot \underbrace{\partial_i f_w}_{\text{in } \Psi_{w=y}}$$

(because Φ be subckt over Ψ $S(\Psi) + S(\Phi)$)

compute $\partial_i f$ by adding constant number of edges.

Computing Partial Derivatives

Open problem: can we also compute second order partial derivatives with only a constant blow-up in ckt size?

non-trivial

Consequence: if yes, we would get that matrix multiplication can be computed in $O(n^2)$ time.

nontrivial

Practice problem: prove this consequence.

Depth Reduction

Theorem ([Valiant, Skyum, Berkowitz, Rackoff 1983])

If a homogeneous polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d can be computed by a circuit of size s, there is a homogeneous circuit Φ of size $\text{poly}(d, s)$ computing p such that:

1. *Φ has alternating levels of sum and product gates*
2. *each product gate $v \in \Phi$ computes the product of five polynomials, each of degree $\leq 2 \cdot \deg(v)/3$*
3. *Sum gates have arbitrary fanin* $\}$ $O(\log s)$ depth fanin 2

In particular, the number of levels in Φ is $O(\log d)$.

The above yields circuit (with fanin 2 for every gate) of depth

$$O(\underbrace{\log d}_{\#\text{levels}}\underbrace{(\log d + \log s)}_{\hookrightarrow \text{ sum gates fanin 2}})$$

Depth Reduction

$NC^1 \subset NC^2 \subset \cdots$

Theorem ([Valiant, Skyum, Berkowitz, Rackoff 1983])

If a homogeneous polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ *of degree* d *can be computed by a circuit of size* s*, there is a homogeneous circuit* Φ *of size* $\mathrm{poly}(d, s)$ *computing* p *such that:*

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3. *Sum gates have arbitrary fanin*

In particular, the number of levels in Φ *is* $O(\log d)$*.*

The above yields circuit (with fanin 2 for every gate) of depth

$$O(\log d(\log d + \log s))$$

Corollary

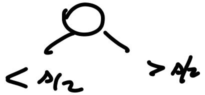
$$VP = VNC^2$$

Depth Reduction

Theorem (Depth Reduction for Formulas)

If $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ is computed by a formula of size s, then $p(x_1, \ldots, x_n)$ can also be computed by a formula of depth $O(\log s)$.

Depth Reduction



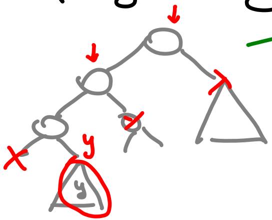
Φ size s

find gate v s.t. $S(\Phi_v)$ satisfies

$$\frac{s}{3} \le S(\Phi_v) \le \frac{2s}{3}$$

$\Phi_{v=y}$

$$f_{v=y} = \underbrace{y \cdot \partial_v f}_{\text{formula}} + f_{v=y}\Big|_{y=0}$$



$$f = f_v \cdot \partial_v f + \overbrace{f_{v=y}\big|_{y=0}}^{s - S(\Phi_v)}$$

Depth Reduction

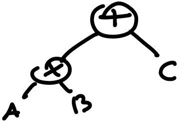
$$\frac{s}{3} \le S(\Phi_v) \le \frac{2s}{3}$$

$$\left(\frac{2}{3}\right)^i s \to O(i)$$

$$i = O(\log s)$$

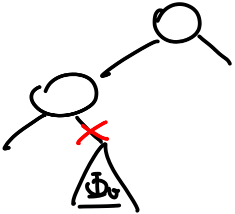
$$\underbrace{s - S(\Phi_v)}_{f_{v=y}|_{y=0}} \le \frac{2s}{3}$$

$$f = \overbrace{f_v}^{\le \frac{2s}{3}} \cdot \underbrace{\partial_v f}_{\le \frac{2s}{3}} + f_{v=y}\big|_{y=0}$$



$$\Phi = A \cdot B + C$$

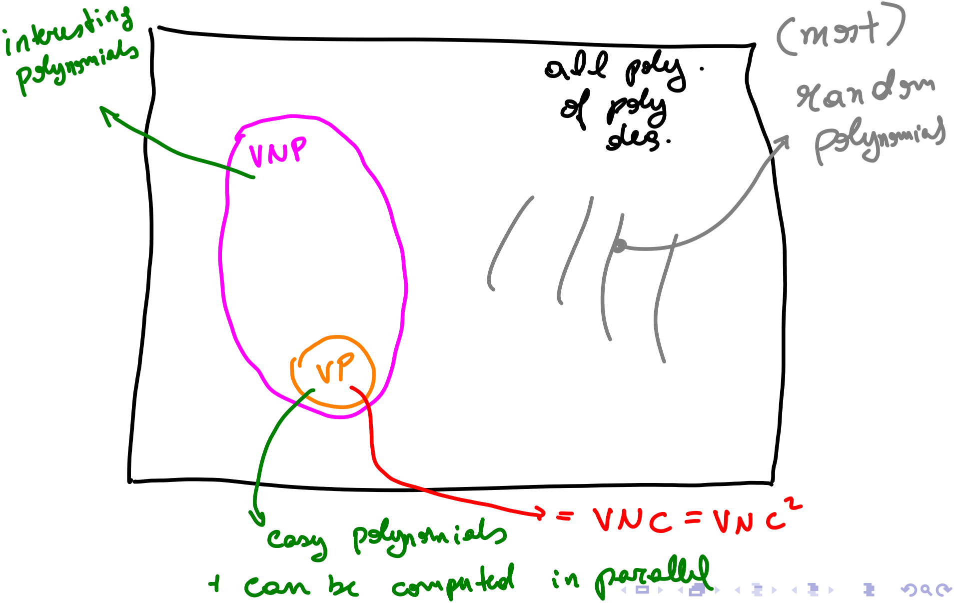
$$s(A),\ s(B),\ s(C) \le \frac{2s}{3}$$



Depth Reduction

$$VP_{formulas} \underset{\supset}{\subset} VNC^1$$

Our world map so far



Conclusion

- Today we learned some additional algebraic complexity classes
  - constant depth circuits (formulas)
  - algebraic branching programs
  - algebraic formulas
- Construction of universal circuit
- Efficient computation of partial derivatives using algebraic circuits
- Depth reduction
- Consequences to algebraic complexity

Acknowledgement

- Lecture based largely on:
  - Excellent survey [Shpilka & Yehudayoff 2010, Chapter 2]

    https://www.nowpublishers.com/article/Details/TCS-039

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