Lecture 2: Algebraic Circuits & Algebraic Complexity

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Overview

- Algebraic Complexity Classes
- Structural Results on Algebraic Circuits
- Conclusion
- Acknowledgements

Complexity Measures in Algebraic Circuits

- *circuit size:* number of edges in the circuit, denoted by $\mathcal{S}(\Phi)$
- *cost of ring elements:* in classical algebraic complexity, there is unit cost for the use of any base ring element
- Sometimes we will add bit complexity of base ring elements

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- *circuit depth:* length of longest direct path from an input to an output
- *constant depth circuits:* for circuits of constant depth, we don't place restriction on the fan-in of an edge.

Examples - Constant Depth Circuits

Algebraic Formulas

- when the computation graph is a tree (i.e., we don't reuse computation) we get an algebraic formula

Algebraic Branching Programs

- polynomials which are projections of the *Iterated Matrix Multiplication* (IMM) polynomial

$$\mathrm{tr}[X_1 X_2 \cdots X_d]$$

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Homogeneous Components

Theorem ([Strassen 1973])

If a polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ *can be computed by a circuit* Φ *of size* $\mathcal{S}(\Phi)$*, then the homogeneous components* $H_0[p], H_1[p], \ldots, H_r[p]$ *can be computed by a circuit of size* $O(r^2 \cdot \mathcal{S}(\Phi))$.

Universal Circuits

Definition

A circuit Φ is called (n, d, s)-*universal*, if the following holds:
If $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$ are homogeneous polynomials of degree d which can be simultaneously computed by a circuit of size s, then there is a circuit Ψ computing $f_1, \ldots, f_n$ with same computation graph as Φ.

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- Φ is (n, d, s)-universal if any circuit Ψ of size $\leq s$ computing homogeneous polynomials of degree d are a projection of Φ
- Normal-homogeneous form:
  - all input gates are labelled by a variable
  - all edges leaving input gates are connected to sum gates
  - all output gates are sum gates
  - alternating sum-product layers
  - fanin of each *product* gate is exactly 2
  - out-degree of each *addition* gate is ≤ 1

Universal Circuits

Theorem ([Raz 2008])

For any integers $s \geq n$ and d, we can construct in time $\mathrm{poly}(s, d)$ *a circuit* Φ *in normal-homogeneous form with at most* $O(s^4 d)$ *nodes that is* (n, d, s)*-universal.*

- For every circuit Ψ, there is a circuit χ in normal homogeneous form computing all polynomials that Ψ computes

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- Every circuit Ψ in normal homogeneous form is a projection of the universal circuit

Computing Partial Derivatives

Theorem ([Baur, Strassen 1983])

If a polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ can be computed by a circuit Φ of size s and depth d, then there is a circuit Ψ of size $O(s)$ and depth $O(d)$ computing (simultaneously) the polynomials $\partial_1 p, \partial_2 p, \ldots, \partial_n p$.

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- Taking partial derivative with respect to a gate

Computing Partial Derivatives

- Induction on circuit size

Computing Partial Derivatives

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Depth Reduction

Theorem ([Valiant, Skyum, Berkowitz, Rackoff 1983])

If a homogeneous polynomial $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ of degree d can be computed by a circuit of size s, there is a homogeneous circuit Φ of size $\mathrm{poly}(d, s)$ computing p such that:

1. *Φ has alternating levels of sum and product gates*
2. *each product gate $v \in \Phi$ computes the product of five polynomials, each of degree $\leq 2 \cdot \deg(v)/3$*
3. *Sum gates have arbitrary fanin*

In particular, the number of levels in Φ is $O(\log d)$.

The above yields circuit (with fanin 2 for every gate) of depth

$$O(\log d(\log d + \log s))$$

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Corollary

$$VP = VNC^2$$

Depth Reduction

Theorem (Depth Reduction for Formulas)

If $p(x_1, \ldots, x_n) \in \mathbb{F}[x_1, \ldots, x_n]$ is computed by a formula of size s, then $p(x_1, \ldots, x_n)$ can also be computed by a formula of depth $O(\log s)$.

Depth Reduction

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Depth Reduction

Our world map so far

Conclusion

- Today we learned some additional algebraic complexity classes
  - constant depth circuits (formulas)
  - algebraic branching programs
  - algebraic formulas
- Construction of universal circuit
- Efficient computation of partial derivatives using algebraic circuits
- Depth reduction
- Consequences to algebraic complexity

Acknowledgement

- Lecture based largely on:
  - Excellent survey [Shpilka & Yehudayoff 2010, Chapter 2]
    https://www.nowpublishers.com/article/Details/TCS-039

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Vermeidung von Divisionen

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