# Lecture 19: Introduction to Convex Algebraic Geometry

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# Overview

- Convexity
- Carathéodory's Theorem
- Positive Polynomials
- Conclusion

# Euclidean Space

- Euclidean space $V$
  1. $V$ is $\mathbb{R}$-vector space
  2. $V$ has positive definite inner product

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

$$||x||^2 = \langle x, x \rangle$$

$$d(x, y) = ||x - y||$$

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  1. $\mathbb{R}^d$ with usual inner product

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- Examples
  1. $\mathbb{R}^d$ with usual inner product
  2. If $D = \binom{n-1+d}{d}$, then $\mathbb{R}^D$ with inner product defined by:

\# monomials of deg $d$ over $n$ variables

$$\langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \rangle = \begin{cases} 0, & \text{if } \mathbf{a} \neq \mathbf{b} \\ \dfrac{\mathbf{a}!}{d!}, & \text{otherwise} \end{cases}$$

$\bar{a} = (a_1, \ldots, a_n)$

$\prod_{i=1}^{n} (a_i!)$

Bombieri inner product on space of homogeneous polynomials of degree $d$.

# Convex sets, convex combinations and convex hulls

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$$y = \alpha_1 x_1 + \cdots + \alpha_m x_m, \quad \text{where} \quad \alpha_i \geq 0, \quad \text{and} \quad \alpha_1 + \cdots + \alpha_m = 1$$

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- Given a set $X \subset V$, the *convex hull* of $X$, denoted $\text{conv}(X)$, is the set of all convex combinations of *finite subsets* of $X$

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$$a = \sum_{i=1}^{m} \alpha_i u_i, \quad \text{and} \quad b = \sum_{i=1}^{n} \beta_i v_i$$

the interval $[a, b]$ satisfies

$$\lambda a + (1-\lambda) b = \lambda \sum_{i=1}^{m} \alpha_i u_i + (1-\lambda) \sum_{i=1}^{n} \beta_i v_i$$

$$\sum_{i=1}^{m} \lambda \alpha_i + \sum_{i=1}^{n} (1-\lambda) \beta_i = \lambda \cdot 1 + (1-\lambda) 1 = 1$$

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$\sum \alpha_i = 1$

$\alpha_i \geq 0$

need to prove that $u \in A$. Can assume $\alpha_i > 0$.

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- Prove by induction on $n$.
- Base case $n = 1$ holds by definition. And case $n = 2$ holds since $A$ is convex.
- Suppose it is true for $n - 1$. Write

$$u = (1 - \alpha_n)w + \alpha_n u_n$$

where $w = \frac{1}{1 - \alpha_n} \cdot (\alpha_1 u_1 + \cdots + \alpha_{n-1} u_{n-1})$

$\beta_i = \frac{\alpha_i}{1 - \alpha_n}$ $\quad w = \sum_{i=1}^{n-1} \beta_i u_i$ $\quad \beta_i \geq 0$ $\quad \sum_{i=1}^{n-1} \beta_i = \frac{\sum^{n-1} \alpha_i}{1 - \alpha_n} = \frac{1 - \alpha_n}{1 - \alpha_n} = 1$

$w \in A$ by hypothesis

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# Gauss-Lucas Theorem

- If $f(z) \in \mathbb{C}[z]$ is a non-constant polynomial and $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ are its roots, seen as elements $\alpha_i = (x_i, y_i) \in \mathbb{R}^2$, we have that the roots of $f'(z)$ are in $\text{conv}(\alpha_1, \ldots, \alpha_m)$.

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- Note that $f(z) = \prod_{i=1}^{m}(z - \alpha_i)$ and thus

$$f'(z) = \sum_{i=1}^{m} \prod_{j \neq i} (z - \alpha_j)$$

product rule

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- If $\beta$ is a root of $f'(z)$, we have ($\beta \neq \alpha_i$)

$$0 = f'(\beta) = \sum_{i=1}^{m} \prod_{j \neq i} (\beta - \alpha_j) \;\Rightarrow\; 0 = \overline{f'(\beta)} = \sum_{i=1}^{m} \prod_{j \neq i} \overline{(\beta - \alpha_j)}$$

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- Multiplying both sides by $f(z)$ we get

$$0 = \sum_{i=1}^{m} (\beta - \alpha_i) \cdot \underbrace{\prod_{j \neq i} |\beta - \alpha_j|^2}_{\gamma_i}$$

$\gamma_i > 0$ $(\beta \neq \alpha_j)$

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$$0 = \sum_{i=1}^{m} \gamma_i (\beta - \alpha_i) \qquad \gamma_i > 0$$

$$\delta_i = \frac{\gamma_i}{\sum_{i=1}^{n} \gamma_i} \qquad 0 = \sum_{i=1}^{m} \delta_i (\beta - \alpha_i) \qquad \sum_{i=1}^{m} \delta_i = 1$$

$$\delta_i > 0$$

$$\beta = \sum_{i=1}^{m} \delta_i \beta = \sum_{i=1}^{m} \delta_i \alpha_i$$

$$\Rightarrow \beta \in \operatorname{conv}(\alpha_1, \ldots, \alpha_m).$$

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- If we have a number $\alpha \in \mathbb{R}$, then $\alpha A$ is a *scaling* (or dilation) of $A$
- A surprising property:

  If $A \in \mathbb{R}^d$ is a compact convex set and $B = (-1/d) \cdot A$, then there is a vector $b \in \mathbb{R}^d$ such that $b + B \subset A$.

# Carathéodory's Theorem

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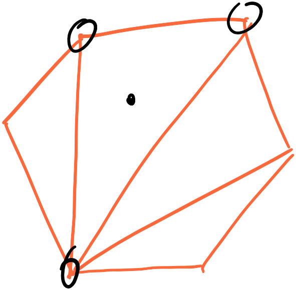
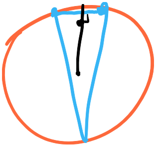


in 2-D we can always write pt as convex comb. 3 pts



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- If $m \leq d+1$ we are done. Otherwise, assume that $m > d+1$ and $\alpha_i > 0$ for all $i \in [m]$. Consider the following system:

$y_i \in \mathbb{R}^d$ $\quad d$

$$\gamma_1 y_1 + \cdots + \gamma_m y_m = 0 \quad \text{and} \quad \gamma_1 + \cdots + \gamma_m = 0$$

homogeneous system: $m$ variables $\gamma_1, \ldots, \gamma_m$; $d+1$ equations

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- Let

$$\tau = \min\{\alpha_i/\gamma_i \mid \gamma_i > 0\} = \alpha_{i_0}/\gamma_{i_0}$$

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- Have $\delta_i := \alpha_i - \tau\gamma_i$ satisfy:

$$\sum \alpha_i - \tau \sum \gamma_i = 1 + 0 = 1$$

$$\delta_{i_0} = 0, \qquad \delta_1 + \cdots + \delta_m = 1, \qquad \delta_i \geq 0$$

$$\delta_i > \alpha_i \geq 0$$

$$\delta_i = \alpha_i - \tau\gamma_i \geq \begin{cases} \gamma_i < 0 \Rightarrow \tau(-\gamma_i) > 0 \\ \gamma_i > 0 \Rightarrow \alpha_i - \tau\gamma_i \geq \alpha_i - \frac{\alpha_i}{\gamma_i} \cdot \gamma_i \geq 0 \end{cases}$$

$$\tau \leq \frac{\alpha_i}{\gamma_i}$$

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$$\delta_{i_0} = 0, \qquad \delta_1 + \cdots + \delta_m = 1, \qquad \delta_i \geq 0$$

- So we got a representation of $x$ as a convex combination of $< m$ points. Iterating this procedure we get our result.

# Convex hull and compactness

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$$S^{d+1} \times \Delta \subset V^{d+1} \times \mathbb{R}^{d+1}$$

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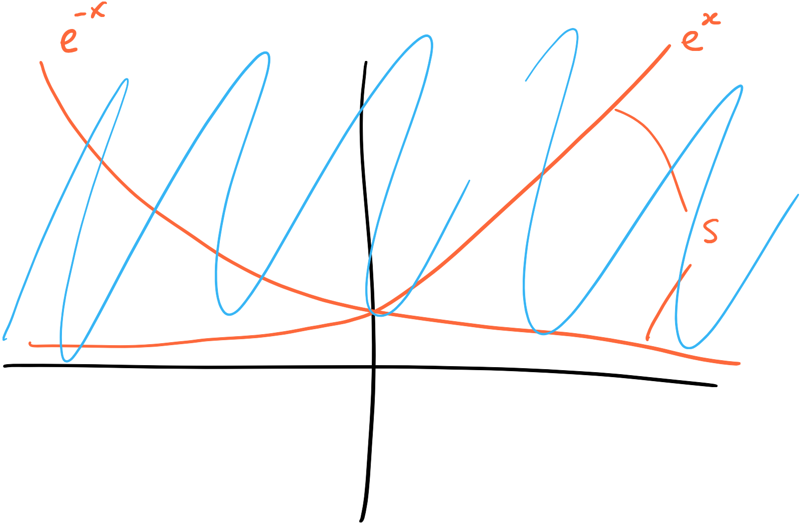
- Thus, the direct product is also compact

$$V^{d+1} \times \mathbb{R}^{d+1} \qquad S^{d+1} \times \Delta$$

- The map $\Phi : S^{d+1} \times \Delta \to \mathbb{R}^d$

$$\Phi(y_0, \ldots, y_d, \alpha_0, \ldots, \alpha_d) = \alpha_0 y_0 + \cdots + \alpha_d y_d$$

  is continuous, and $Im(\Phi) = \mathrm{conv}(S)$ by Carathéodory.
  Thus, $\mathrm{conv}(S)$ is also compact.



$e^{-x}$
$e^{x}$
S

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- Orthogonal basis of $H(n, 2d)$ given by monomials.

$$D = \binom{n-1+d}{d}$$ dimension of Euclidean space $H(n,2d)$

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- The set of *positive polynomials* $PD(n, 2d) \subset H(n, 2d)$ is the set

$$PD(n, 2d) := \{p(\mathbf{x}) \in H(n, 2d) \ | \ p(\mathbf{a}) > 0, \ \forall \mathbf{a} \in \mathbb{R}^n\}$$

$\bar{a} \neq \bar{0}$

$d = 1 \quad n = 2$

$x^2 + y^2$

$(x+y)^2 + (x+2y)^2 + (x+3y)^2$

$\in PD(2,2)$

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- The set of *non-negative polynomials* PSD$(n, 2d) \subset H(n, 2d)$ is the set

$$PSD(n, 2d) := \{p(\mathbf{x}) \in H(n, 2d) \;|\; p(\mathbf{a}) \geq 0, \;\forall \mathbf{a} \in \mathbb{R}^n\}$$

$PD(n, 2) \leftrightarrow$ positive definite matrices

$PSD(n, 2) \leftrightarrow$ positive semidefinite matrices

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$$= (x_1^2 + x_2^2 + \cdots + x_n^2)^d$$

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  3. $q(\mathbf{x})$ invariant and $q(\mathbf{y}) = 0$ implies $q(\mathbf{x}) \equiv 0$

$\alpha \in \mathbb{R}^n$

$\|\alpha\| = \|\bar{y}\|$

$\exists\ U_\alpha \bar{y} = \alpha$

$q(\alpha) = 0$ for all $\alpha \in S^{n-1} \cdot \|\bar{y}\|$

$\Rightarrow q(\bar{x}) = 0$

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  4. Thus, $p(\mathbf{x}) = \gamma \cdot \|\mathbf{x}\|_2^{2d}$

# An Interesting Identity

- There exist vectors $\mathbf{c}_1, \ldots, \mathbf{c}_m \in \mathbb{R}^n$ such that

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- Let us consider unit sphere

$$S^{n-1} := \{\mathbf{c} \in \mathbb{R}^n \mid \|\mathbf{c}\|_2 = 1\}$$

the set

$$K = \operatorname{conv}\left(\langle \mathbf{c}, \mathbf{x} \rangle^{2d} \mid \mathbf{c} \in S^{n-1}\right)$$

is compact, by our corollary of Carathéodory.

$S^{n-1} \longrightarrow H(n, 2d)$

$\bar{c} \mapsto \langle \bar{c}, \bar{x} \rangle^{2d}$

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$m = D + 1$

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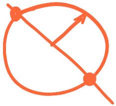
  $\gamma > 0$ because given $\mathbf{a} \in S^{n-1}$, $\langle \mathbf{c}, \mathbf{a} \rangle^{2d} = 0$ for a set of measure zero.

$p(\bar{a})$

$\langle \bar{c}, \bar{a} \rangle = 0$ set of measure 0



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- We can approximate the integral above by finite Riemann sum

$$p(\mathbf{x}) \approx \frac{1}{N} \cdot \sum_{i=1}^{N} \langle \mathbf{c}_i, \mathbf{x} \rangle^{2d}$$

$q_N(x) \in K$

thus $p$ is in the closure of $K$.

$\{q_N\}_N \to p \Rightarrow p \in \overline{K} \Rightarrow p \in K.$

$\Rightarrow \gamma \|\bar{x}\|_2^{d} \in K$

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- $K$ compact (thus closed) implies that $p \in K$, and by Carathéodory we have

$$\|\mathbf{x}\|_2^{2d} = \sum_{i=1}^{D+1} \langle \mathbf{c}_i, \mathbf{x} \rangle^{2d}$$

$\bar{e}_i = \frac{\bar{c}_i}{\sqrt{\gamma}}$

# Application: uniform sum-of-squares

- Hilbert's 17th problem: given a polynomial in $PSD(n, 2d)$, can it be written as a sum of quotients of square polynomials?

Emil Artin 1927

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  Is there a polynomial $p(\mathbf{x}) \in H(n, 2d)$ such that given *any* polynomial $q(\mathbf{x}) \in PSD(n, 2d)$, there is $N \in \mathbb{N}$ such that

  ↖ may depend on q

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  is a sum of square polynomials?
- Pólya proved that the answer is YES!
- Pólya & Reznick's Theorem:

  Can take $p(\mathbf{x}) = \|\mathbf{x}\|_2^{2d}$

# Conclusion

- Today we started our study of convexity from an algebraic perspective
- Learned some structural results on convex sets

  Carathéodory's theorem

- Saw applications of Carathéodory's theorem to prove a Waring-type result

  Writing certain polynomial as sum of powers of linear forms.

- Application to writing positive polynomials as sum of squares of a particular rational function

  Uniform version of Hilbert's 17th problem.

# References I

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