Modules

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Given a ring S, an S module $\underline{M}$ is an abelian group with a "scalar multiplication" action of S with the following properties.

(i) $(r+s)\cdot m = r\cdot m + sm$

(ii) $r\cdot(m+m') = r\cdot m + r\cdot m'$

(iii) $(rs)\cdot m = r\cdot(s\cdot m)$

(iv) $1\cdot m = m$

Equivalently, there is a ring map from S to $End(M)$.

Examples.

(i) If S is a field, then modules are vector spaces.

(ii) If I is an ideal of S, then I is an S-module

here, $r.a = ra$

(iii) If I is an ideal, then S/I is an S. module.

here, $r.[a] = [ra]$

(iv) $S^n = S \oplus S \oplus \cdots \oplus S$ is an S module.

an element of $S^{\oplus n} = (r_1, \ldots\ldots r_n)$

here, $r.(r_1, r_2, \ldots r_n) = (rr_1, \ldots\ldots rr_n)$.

Let $S = k[x,y,z]$. Let $M = (xy, xz)$.

$$z.(xy) - y.(xz) = 0$$

We say M is generated by $m_1, \ldots\ldots, m_n$ if every $m \in M$ can be written as

$$m = \sum r_i m_i \quad \text{for } r_i \in S.$$

In other words, every element of M is a S-linear combination of the m_i.

Suppose S is a polynomial ring in finitely many variables.

If M is finitely generated, then every submodule of M is finitely generated.

A special case is the fact that ideals of polynomial rings are finitely generated.

$$S = k[x_1, x_2, \ldots \quad]$$

S is a free S module

$(x_1, x_2, \ldots \quad .)$ is not finitely generated.

The syzygy theorem

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Suppose M is a module over the polynomial ring
$S = k[x_1, \dots x_r]$.
Suppose M is generated by $m_1, \dots, m_n$.

A syzygy is a set of elements $r_1, \dots r_n$ that satisfy $\sum r_i m_i = 0$ $\quad r_i \in S$

We have a surjective map

$$\phi_0 : S^{\oplus n} \longrightarrow M$$

that maps basis e_i to m_i.
The kernel of this map contains elements of the form
$\sum r_i e_i$ that satisfy $\sum r_i m_i = 0$
In other works, the kernel is exactly the syzygies.

The syzygies are a submodule: if $\sum r_i e_i$ and $\sum r_i' e_i$
$\sum (r_i + r_i') m_i = \sum r_i m_i + r_i' m_i = 0$

This kernel is itself a finitely generated submodule of $S^{\oplus n}$

Suppose it has generators $l_1, \dots l_{k_0}$.
We can then define map
$$\phi_1 : S^{\oplus k_0} \longrightarrow S^{\oplus n}$$
that sends e_i to l_i. This map is surjective on to the kernel of ϕ_0.

The map ϕ_1 itself has a kernel which is finitely generated.
We can therefore repeat the above to get a sequence of

The map ϕ_1 itself has a kernel which is finitely generated. We can therefore repeat the above to get a sequence of maps

$$\cdots \xrightarrow{\phi_4} S^{\oplus k_2} \xrightarrow{\phi_3} S^{\oplus k_1} \xrightarrow{\phi_2} S^{\oplus k_0} \xrightarrow{\phi_1} S^{\oplus n} \xrightarrow{\phi_0} M$$

where each ϕ_k is surjective on to the kernel of ϕ_{k-1}.

This is called a free resolution of M.

For example, let $M = (xy, xz)$. Then we have resolution

$$0 \xrightarrow{\phi_2} S \xrightarrow[e_1 \to ze_1 - ye_2]{\phi_1} S^{\oplus 2} \xrightarrow[\substack{e_1 \to xy \\ e_2 \to xz}]{\phi_0} M$$

$\ker(\phi_0)$? $\quad ze_1 - ye_2 \in \ker(\phi_0)$

The length of the above resolution is $\underline{1}$.

The syzygy theorem states that every finitely generated module has length at most r, where r is the number of variables.

If $M = S^{\oplus n}$

$$0 \to S^{\oplus n} \to M$$

$S = k[x,y] \qquad M = S/(x^2 - x, xy, y^2 - y)$

$$0 \to S^2 \to S^3 \to S \to M \to 0$$

Special case, $r=0$. Here, M is a vector space, and is itself free. If M has dimension n, then we have map

$$\phi_0 : S^n \to M \qquad (\text{when } S = k)$$

which is an isomorphism. Therefore M has free resolution of length 0.

Monomials in free modules

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$S = k[x_1, \dots, x_r]$.

Let $F = S \oplus S \oplus \dots \oplus S$ be a free module.

We use $\bar{x}^{\bar{a}}$ to denote the monomial $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ in S

A monomial of F is an element of the form
$m = \bar{x}^{\bar{a}} e_i$.

Suppose also $m' = \bar{x}^{\bar{a}'} e_j$

We say m is divisible by m' if $i = j$ and
$\bar{x}^{\bar{a}}$ is divisible by $\bar{x}^{\bar{a}'}$.

In this case, the quotient is $\bar{x}^{\bar{a}'} / \bar{x}^{\bar{a}} \in S$

If $i = j$, we can also define the GCD, LCM of
m and m' as

$$GCD(m, m') = GCD(\bar{x}^{\bar{a}}, \bar{x}^{\bar{a}'}) \cdot e_i$$

$$LCM(m, m') = LCM(\bar{x}^{\bar{a}}, \bar{x}^{\bar{a}'}) \cdot e_i$$

$m = (x^2 y, 0, 0)$ $\quad m' = (yz, 0, 0)$ $\quad$ then $m \nmid m'$,
$lcm(m, m') = (x^2 yz, 0, 0)$ $\quad (= x^2 yz \cdot e_1)$
$gcd(m, m') = y \cdot e_1$.
If $m'' = (0, x^2 y, 0)$

A monomial submodule of F is a submodule generated by monomials. This extends the notion of monomial ideals.

The syzygies of a Monomial Submodule.

Let $m_1, \ldots, m_n$ be monomials of F

Let $m_{ij} = \dfrac{m_i}{GCD(m_i, m_j)}$ if m_i, m_j are in the same component.

We claim that $\sigma_{ij} = m_{ji} e_i - m_{ij} e_j$ generate the Syzygies.

Recall the definition of $S(f,g)$

$$= \frac{LT(g)}{GCD(LT(g), LT(f))} \cdot f - \frac{LT(f) \cdot g}{GCD(LT(g), LT(f))}$$

__Proof:__

Suppose $\sum r_i m_i = 0$ is a relation.
We can rewrite this as

$$\sum_{n \text{ a monomial in } F} \sum_i (r_i m_i)_n$$

Suppose $m_1 = (xy, 0)$
$r_1 = 2x + z$
If $n = (x^2 y, 0)$
then $(r_1 m_1)_n = (2x^2 y, 0)$
$= 2 . n$

where $(r_i m_i)_n$ is the part of $r_i m_i$ that is a k-multiple of n.

The different summands do not interact, and each $\sum_i (r_i m_i)_n = 0$

Assume therefore that the syzygy is
$\sigma = \sum r_i e_i$, such that $r_i m_i$ is a $\mathbb{C}$ - multiple
of n for every i. (n is a monomial of F)

We now induct on the number of nonzero r_i.
If the syzygy is nonzero, then there are two
nonzero r_i, say r_i and r_j.
Then m_i and m_j divide n, and r_i is
divisible by m_{ji}.

$$r_i m_i = n \qquad r_j m_j = n \quad \Rightarrow \quad m_i | n \text{ and } m_j | n.$$

$$\Rightarrow LCM(m_i, m_j) \mid n.$$

$$\Rightarrow \frac{m_i \cdot m_j}{GCD(m_i, m_j)} \mid n = r_i m_i \quad \Rightarrow \quad m_{ji} | r_i.$$

$$m_{ji} = \frac{m_j}{GCD(m_i, m_j)}$$

We can now subtract $\frac{r_i}{m_{ji}} \sigma_{ij}$ from
σ to get a syzygy with fewer terms.

$$\frac{r_i}{m_{ji}} \sigma_{ij} = r_i e_i - \frac{m_{ij} \cdot r_i}{m_{ji}} e_j$$

The definition of a monomial order carries over from the case of S.

Examples: Suppose $>_S$ is a monomial order on monomials of S.

We define an order on $F = S^{\oplus n}$ extending $>_S$ as follows

$$m = \bar{x}^{\bar{a}} e_i > \bar{x}^{\bar{b}} e_j = m'$$

if $i < j$ or $i = j$ and $\bar{x}^{\bar{a}} >_S \bar{x}^{\bar{b}}$.

We can define other orders, but the above is what we will use.

We can also define the division algorithm, remainders and standard expressions.

Suppose $g_1, \dots g_n, f$ are elements of F.

We can write $f = \sum r_i g_i + f'$ with $r_i \in S$, $f' \in F$

Such that none of the monomials in f' divide a leading monomial of any g_i.

The f' is called a remainder, and the expression is called a Standard expression.

We can find the standard expression using the division algorithm.

Suppose I is an ideal, generated by $r_1, \dots r_n$.
$\{r_i\}$ is a grobner basis iff $LT(I) = (LT(r_1), \dots LT(r_n))$

$LT(I) = \{a \mid a \text{ is the leading term of some } b \in I\}$.

<u>Grobner basis</u>: Given $g_1, \dots g_n$ generators of a module M, we say $g_1, \dots g_n$ is a grobner basis if the leading terms of $g_1, \dots g_n$ generate the module of leading terms $LT(M)$.

Recall that $LT(M)$ is the module that consists of the leading terms of all elements of M.

<u>Buchbergers criterion:</u>

Recall that the criterion for ideals was that the remainder of every S-polynomial was 0.

Essentially the same is true here. We state it for completeness.

Let $g_1, \dots g_n$ be nonzero elements of F.
Let $\phi : S^{\oplus n} \to F$ be the map

Let $g_1, \ldots, g_n$ be nonzero elements of F.

Let $\phi : S^{\oplus n} \to F$ be the map

$$\phi(e_i) = g_i.$$

For every i, j such that $LT(g_i)$ and $LT(g_j)$ are in the same component, define

$$m_{ij} = LT(g_i) \,/\, GCD(LT(g_i), LT(g_j))$$

Set $\sigma_{ij} = m_{ji} e_i - m_{ij} e_j$ for $i < j$

(Note that σ_{ij} generate the Syzygies of the $LT(g_i)$).

Set f_k^{ij} s.t. $\quad (f_k^{ij} \in S)$

$$\boxed{m_{ji} g_i - m_{ij} g_j} = \sum_u f_u^{ij} g_u + h^{ij}$$ is a standard representation.

Then the g_i are a Grobner basis iff $h^{ij} = 0$ for all i, j.

A basis for the syzygies

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Let the notation be the same as before, and let $g_1, \dots, g_n$ be a Grobner basis, so $h^{ij} = 0$ for all i, j

For every $i<j$, let $\tau_{ij} = m_{ji} e_i - m_{ij} e_j - \sum_u f_u^{ij} e_u$ $\quad (\tau_{ij}(g_1, \dots, g_n) = h^{ij})$

so that τ_{ij} is a syzygy.

Define a monomial order on $S^{\oplus n}$ as follows.

$$\bar{x}^{\bar{a}} e_i > \bar{x}^{\bar{b}} e_j \text{ iff}$$

$$\phi_0 : S^{\oplus n} \xrightarrow{u} F, \quad e_i \mapsto g_i$$

$$\operatorname{Ker}(\phi_0) \subseteq \underline{\underline{S^{\oplus n}}}$$

$LT(\bar{x}^{\bar{a}} g_i) > LT(\bar{x}^{\bar{b}} g_j)$ in F or

$LT(\bar{x}^{\bar{a}} g_i) = LT(\bar{x}^{\bar{b}} g_j)$ and $i < j$.

Then the τ_{ij} are a Grobner basis for the syzygies with the above monomial order.

Proof: We first show that $LT(\tau_{ij}) = m_{ji} e_i$.

By construction,

$$LT(m_{ji} g_i) = LT(m_{ij} g_j) > LT(f_u^{ij} g_u)$$

Since $i<j$ we have $m_{ji} e_i > m_{ij} e_j$.

Now let $\tau = \sum f_v e_v$ be any syzygy.

We show that the leading term of τ is divisible by $m_{ju} e_u$ for some u, j.

Set $n_v e_v = LT(f_v e_v)$. Each of these are in a different component, so they cannot cancel.

Therefore, $LT(\tau) = n_u e_u$ for some fixed u.

This means that $LT(n_u g_u) \geq LT(n_v e_v)$ for every v.

Let $\sigma = \sum' n_w e_w$ be the sum over indices s.t.

$LT(n_u g_u) = LT(n_w g_w) = m$.

Every index must be $\geq u$, since $n_u e_u = LT(\tau)$.

The element σ is a syzygy of $LT(g_u), LT(g_{u+1}) \dots, LT(g_n)$

The element σ is a syzygy of $LT(g_u), LT(g_{u+1}) \dots, LT(g_n)$
It is therefore generated by σ_{ij} with $i, j \geq u$.
Since $n_u e_u \neq 0$, there are σ_{uj} with nonzero coefficient.

We have $\sigma_{uj} = m_{ju} e_u - m_{uj} e_j$.

Therefore, n_u is in the ideal generated by the m_{ju}.

$\square$

The above gives us a method of getting the syzygies of a Grobner basis. We use this to prove the main theorem.

<u>Corollary of the above:</u>

Suppose $g_1, \dots, g_n \in F$ are arranged so that whenever $LT(g_i)$ and $LT(g_j)$ involve the same component of F, say $LT(g_i) = \bar{x}^{\bar{a}} e$ and $LT(g_j) = \bar{x}^{\bar{b}} e$,

$$i < j \Rightarrow \bar{x}^{\bar{a}} >_{lex} \bar{x}^{\bar{b}}.$$

If the leading terms of $g_1, \dots g_n$ do not involve $x_1, \dots, x_s$ then the leading terms of τ_{ij} do not involve $x_1, \dots, x_s, x_{s+1}$.

<u>Proof:</u> The leading term of τ_{ij} is $m_{ji} e_i$, where

$$m_{ji} = \frac{m_j}{GCD(m_i, m_j)}. \quad \text{where } m_i = LT(g_i)$$

The degree in x_{s+1} of m_i is greater than or equal to that of m_j, since $m_i > m_j$ in lex order.
therefore, m_{ji} does not involve x_{s+1}.

<u>Proof of the syzygy theorem:</u>

Let M be an arbitrary module of S, generated by $a_1, \dots a_{n_0}$.

$$\phi_0 : S^{\oplus n_0} \xrightarrow[e_i \to a_i]{} M$$

Let $K_0 = Ker(\phi_0)$.

$e_i \to a_i$

Let $K_0 = \operatorname{Ker}(\phi_0)$.

K_0 is a submodule of a free module. Let $g_1, \dots g_{n_0}$ a grobner basis for K_0.

Arrange $g_1, \dots g_{n_0}$ to satisfy the assumption of prev. corollary

$$S^{\oplus n_1} \xrightarrow[e_i \to g_i]{\phi_1} S^{\oplus n_0} \xrightarrow{\phi_0} M$$

Let τ_{ij} be the grobner basis for $\operatorname{Ker}(\phi_1)$ = Syzygies of $g_1, \dots g_{n_1}$

The $LT(\tau_{ij})$ do not involve x_1. (by prev corr. with $s=0$).

Arrange τ_{ij} to satisfy the criterion of the prev corollary.

Suppose there are n_2 many τ_{ij}

$$S^{\oplus n_2} \xrightarrow[\text{map to } \tau_{ij}]{\phi_2} S^{\oplus n_1} \xrightarrow{\phi_1} .$$

Let η_{ij} be the generators of $\operatorname{Ker}(\phi_2)$

$LT(\eta_{ij})$ do not involve x_1 or x_2

We keep doing this.

After $r-1$ steps, we get

$$\phi_{r-1} : S^{\oplus n_{r-1}} \longrightarrow S^{\oplus n_{r-2}}$$

Let $h_1, \dots h_{r_n}$ be a grobner basis for $\operatorname{Ker}(\phi_{r-1})$.

$LT(h_i)$ only involve x_r.

$LT(h_i) = x_r^{l_i} \cdot e_{u_i}$.

We can assume that all the u_i are distinct, by reducing the basis.

What are the syzygies of $h_1, \dots h_{r_n}$?

there are none.

Therefore, we have the maps

$$0 \longrightarrow S^{\oplus n_r} \xrightarrow[e_i \to k_i]{} S^{\oplus n_{r-1}} \xrightarrow{\phi_{r-1}} \cdots\cdots \longrightarrow S^{\oplus n_0} \xrightarrow{\phi_0} M \longrightarrow 0$$

which is the required resolution.