Lecture 14: Introduction to Geometric Invariant Theory

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March 3, 2021

Overview

- Group Actions on Vector Spaces, Orbits & Orbit Closures
- Geometric Questions
- Conclusion

Group Actions

- Let G be a nice[1] group and V be a $\mathbb{C}$-vector space

[1] The definition of nice is a bit technical, so we will stick to finite groups and $\mathbb{SL}(n)$

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- Let G be a nice[1] group and V be a $\mathbb{C}$-vector space
- G acts *linearly* on V if

$$g \circ (\alpha u + \beta v) = \alpha(g \circ u) + \beta(g \circ v)$$

$$(gh) \circ u = g \circ (h \circ u)$$

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- Examples:

finite
1. $G = S_n,\ V = \mathbb{C}^n$ — permuting coordinates
2. $G = A_n,\ V = \mathbb{C}^n$ — permuting coordinates

continuous
3. $G = \mathbb{SL}(2),\ V = \mathbb{C}^d$ — linear transformations of curves
4. $G = \mathbb{SL}(n),\ V = \mathrm{Mat}(n)$ — left multiplication
5. $G = \mathbb{GL}(n),\ V = \mathrm{Mat}(n)$ — conjugation
6. $G = \mathbb{ST}(n) \times \mathbb{ST}(n),\ V = \mathrm{Mat}(n)$ — row/column scaling

finite
7. $G = S_n,\ V = \mathbb{C}^{\binom{n}{2}}$ — graph isomorphism

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Group Orbits

- Given an element $u \in V$, its *orbit* is defined by

$$\mathcal{O}_u := \{\underbrace{g \circ u}_{\text{can be reached from } u \text{ by action of } G} \mid g \in G\}$$

1. $G = S_n$, $V = \mathbb{C}^n$ permuting coordinates

Permutation of coordinates.

$$\mathcal{O}_{e_1} = \{e_1, e_2, \dots, e_n\}$$

$$\mathcal{O}_{e_1+e_2} = \{e_i + e_j \mid i \neq j \in [n]\}$$

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Permutation of coordinates.

2. $G = \mathbb{SL}(2)$, $V = \mathbb{C}^{d+1}$ change of coordinates

Linear transformations of roots.

$$P(x,y) = \sum_{i=0}^{d} p_i x^i y^{d-i} \longleftrightarrow (p_0, \dots, p_d) \in \mathbb{C}^{d+1}$$

$$= p_0 \prod_{i=1}^{d} (x - \alpha_i y) \qquad \{[\alpha_i : 1]\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

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3. $G = \mathbb{SL}(n),\ V = \mathrm{Mat}(n)$ left multiplication

   Same rank? (No column exchange)

$A \cdot X$ $\qquad \mathcal{O}_X$

$$X_1 = \begin{array}{c} \downarrow{=0} \\ \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \end{array} \qquad X_2 = \begin{array}{c} \downarrow{\neq 0} \\ \begin{pmatrix} 0 & 0 & 0 & \\ 1 & 0 & & \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \\ 0 & 0 & & & 1 \end{pmatrix} \end{array}$$

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4. $G = \mathbb{GL}(n),\ V = \mathrm{Mat}(n)$ conjugation

   Same eigenvalues? (Diagonalizable vs Jordan blocks)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{\text{not}} \text{ diagonalizable}$$

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5. $G = \mathbb{ST}(n) \times \mathbb{ST}(n),\ V = \mathrm{Mat}(n)$ row/column scaling

   Matrix scaling. (Orbits more complex.)

Orbit Closure

V inner product space $\langle \, , \, \rangle$

- Given an element $u \in V$, its *orbit closure* is defined by

$$\overline{\mathcal{O}}_u := \{ g \circ u \mid g \in G \} \cup \underbrace{\text{limit points}}_{\text{Euclidean topology}}$$

$$\begin{pmatrix} x_1 & \\ & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 & \\ & c_2 \end{pmatrix}$$

$$\begin{pmatrix} \epsilon & \\ & 1/\epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\epsilon & \\ & \epsilon \end{pmatrix} = \begin{pmatrix} 1 & \boxed{\epsilon^2} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \notin \mathcal{O}_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \in \overline{\mathcal{O}}_{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$$

Orbit Closure

- Given an element $u \in V$, its *orbit closure* is defined by

$$Z(f_1, \ldots, f_t) = \quad \overline{\mathcal{O}}_u := \{g \circ u \mid g \in G\} \cup \text{ limit points}$$

- Limit points either with respect to Euclidean or Zariski Topologies.

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Permutation of coordinates.

orbits here are already closed

(finite sets of points)

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   Linear transformations of roots.

$d = 2$ $\quad P(x,y) = ax^2 + bxy + cy^2$

$\boxed{(x-y)^2}$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & +1 \\ 0 & 1 \end{pmatrix}}_{A}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad x \mapsto x+y, \quad y \mapsto y$$

$$(x-y)^2 \xmapsto{A} x^2 \xleftrightarrow{B} \epsilon^2 x^2 \to 0$$

$$B_\epsilon = \begin{pmatrix} \epsilon & \\ & 1/\epsilon \end{pmatrix} \in SL(2)$$

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3. $G = \mathbb{SL}(n),\ V = \mathrm{Mat}(n)$ left multiplication

   Same rank? (No column exchange)

any matrix that has rank not full (singular matrices) have 0 in orbit closure

$$X \xrightarrow{A} AX = \begin{pmatrix} * \\ \hline 0\ 0 \cdots 0 \end{pmatrix} \xrightarrow{(\epsilon\ \epsilon \cdots \epsilon\ 1/\epsilon^{n})} \begin{pmatrix} \epsilon * \\ \hline 0\ 0 \cdots 0 \end{pmatrix}$$

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$$\begin{pmatrix} \epsilon & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\epsilon & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Nullcone, Orbit Closure Intersection, Orbit Closure Containment

- G acts linearly on V Hermitian inner product structure $V = \mathbb{C}^N$
- *Orbit Closure Intersection*: Given two points $u, w \in V$, do their orbit closures intersect?

$$\overline{\mathcal{O}}_u \cap \overline{\mathcal{O}}_w \neq \emptyset?$$

Orbit intersection $\Leftrightarrow$ orbits are the same

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$$0 \in \overline{\mathcal{O}}_u?$$

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Hilbert '1893

- Null-cone problem has its name from the definition that the *nullcone* is the set of elements that have zero in their orbit closure.

$$\underbrace{\mathcal{N} = \{u \in V \mid 0 \in \overline{\mathcal{O}}_u\}}_{\text{null cone}}$$

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Closure

$$\overline{\mathcal{O}}_w \subset \overline{\mathcal{O}}_u?$$

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$$\overline{\mathcal{O}}_w \subset \overline{\mathcal{O}}_w?$$

- Given $u \in V$, is its orbit closed?

$$\mathcal{O}_u = \overline{\mathcal{O}}_u?$$

Null Cone Problems and Applications

1. $G = \mathbb{SL}(2)$, $V = \mathbb{C}^{d+1}$ change of coordinates

   Does the polynomial have a root of multiplicity $> d/2$?

Null cone = { polynomials with really high multiplicity }

Null Cone Problems and Applications

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2. $G = \mathbb{SL}(n)$, $V = \mathrm{Mat}(n)$ left multiplication

   Singular Matrices

invariant : $\det(X)$ $\boxed{\text{Continuous}}$

Y is singular $\xrightarrow{A}$ $AY = \begin{pmatrix} * \\ 0 \cdots 0 \end{pmatrix}$

$\Rightarrow 0 \in \overline{\mathcal{O}_Y}$

Y is not singular $\boxed{\det(Y) \neq 0}$

$\| A_\epsilon Y \|_F < \epsilon \Rightarrow \dfrac{\det(A_\epsilon Y)}{\det(Y)} \leq n!\,\epsilon^n$ Cannot happen!

Null Cone Problems and Applications

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

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   ~~*Zero Matrix*~~ Nilpotent matrices

Continous

invariants : $\boxed{\mathrm{tr}(X)}, \boxed{\mathrm{tr}(X^2)}, \ldots, \boxed{\mathrm{tr}(X^n)}$

$\sum \lambda_i(X) \qquad \sum \lambda_i^2(X) \qquad \ldots \sum \lambda_i^n(X)$

$\det(tI - X) = \prod (t - \lambda_i(X))$

$A_\epsilon X A_\epsilon^{-1} \to 0 \implies$ all eigenvalues of X had to be $0 \implies X$ nilpotent

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4. $G = \mathbb{ST}(n) \times \mathbb{ST}(n),\ V = \text{Mat}(n)$ — row/column scaling

   Graphs without bipartite matching.

invariants: permutation monomials (matching) — Continuous

if graph has perfect matching some invariant doesn't vanish $\Rightarrow$ matrix cannot go to zero

if G no perfect matching

$$\begin{pmatrix} \epsilon^2 & & \\ & 1/\epsilon & \\ & & 1/\epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \epsilon^2 & & \\ & 1/\epsilon & \\ & & 1/\epsilon \end{pmatrix}$$

$$\begin{pmatrix} \epsilon^4 & \epsilon & \epsilon \\ \epsilon & 0 & 0 \\ \epsilon & 0 & 0 \end{pmatrix}$$

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5. $G = \mathbb{SL}(n) \times \mathbb{SL}(n),\ V = \mathrm{Mat}(n)^m$ operator scaling

   Word-problem for free skew fields, Rational Identity Testing, Brascamp-Lieb inequalities.

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6. How do we know this?

Orbit Problems and Invariant Polynomials

- **[Hilbert 1893]**: *Nullcone* is the zero set of non-constant, homogeneous *invariant polynomials*.

$$\mathbb{C}[\bar{x}]^G = \mathbb{C} \oplus \underbrace{\mathbb{C}[\bar{x}]^G_1 \oplus \cdots}_{I}$$

$$\mathcal{N} = Z(I)$$

Orbit Problems and Invariant Polynomials

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- **[Hilbert-Mumford]**: orbit closure intersection

$$\overline{\mathcal{O}}_u \cap \overline{\mathcal{O}}_w \neq \emptyset \Leftrightarrow \underline{p(u)} = \underline{p(w)} \ \forall p \in \mathbb{C}[V]^G$$

orbit closures don't intersect

$\Updownarrow$

$\exists p$ invariant s.t. $p(u) \neq p(w)$

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   Discriminants (and more)

Jerzy Weyman 1987

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2. $G = \mathbb{SL}(n),\ V = \mathrm{Mat}(n)$ left multiplication

   Determinant

$$\mathbb{C}[X]^G = \mathbb{C}[\underline{\det(X)}]$$

$$\mathcal{N} = Z(\det(X))$$

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   Trace polynomials.

trace polynomials invariant + Cayley-Hamilton $\boxed{A^n = 0}$

$\det(tI - X) = t^n$

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   Matching/Permutation monomials.

What about Orbit Closure Containment?

- Orbit closure containment much harder problem
- $\overline{\text{VP}}$ vs $\overline{\text{VNP}}$ question

Orbit closure containment is NP-hard problem Bläser et al.

An Optimization View on Nullcone

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- When our vector space have an inner product, motivates the following optimization question:

$$u \longmapsto \qquad \inf_{g \in G} \|g \circ u\|_2 \quad = 0 \ ?$$

An Optimization View on Nullcone

- Note that with the nullcone, we want to know whether 0 in the orbit closure
- When our vector space have an inner product, motivates the following optimization question:

$$\inf_{g \in G} \|g \circ u\|_2$$

- Optimization is over the group elements. Geometry determined by the geometry of the group

Hilbert-Mumford Semistability Theorem

- *Nullcone*: the set of elements that have zero in their orbit closure.

$$\mathcal{N} = \{u \in V \mid 0 \in \overline{\mathcal{O}}_u\}$$

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- 1-parameter subgroups (1-PSG):

$$\phi : \mathbb{C}^* \to G$$

$$t \longmapsto g(t)$$

$SL(2)$

$$t \longmapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$$

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- **[Hilbert-Mumford]**: an element $u \in V$ is in the nullcone if, and only if, there is a 1-PSG which drives u to zero.

Today we will not prove this for two actions:

1. $\mathbb{ST}(n)$ action on $\mathbb{C}^N$
2. $\mathbb{SL}(n)$ action on $\mathbb{C}^{n \times m}$ by left-multiplication.

observe that we proved this
- matrix scaling
- left multiplication

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- $\mathbb{SU}(n) \leftarrow$ maximal compact subgroup

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- 1-parameter subgroups (1-PSG):

$$\phi : \mathbb{C}^* \to G$$

- **[Hilbert-Mumford]**: an element $u \in V$ is in the nullcone if, and only if, there is a 1-PSG which drives u to zero.

Today we will prove this for two actions:

1. $\mathbb{ST}(n)$ action on $\mathbb{C}^N$
2. $\mathbb{SL}(n)$ action on $\mathbb{C}^{n \times m}$ by left-multiplication.

- Note that $\mathbb{SL}(n) = \mathbb{SU}(n) \times \mathbb{ST}(n) \times \mathbb{SU}(n)$
- $\mathbb{SU}(n) \leftarrow$ maximal compact subgroup
- $\mathbb{ST}(n) \leftarrow$ maximal torus

Hilbert-Mumford Semistability

Left multiplication:

$$X \xmapsto[\text{row operations}]{A} AX = \begin{pmatrix} * \\ \hline 0 \; 0 \cdots 0 \end{pmatrix}$$

$$\underbrace{A^{-1}\underbrace{\begin{pmatrix} \epsilon & & & & & \\ & \epsilon & & & & \\ & & \epsilon & & & \\ & & & \ddots & & \\ & & & & \epsilon & \\ & & & & & 1/\epsilon^{n-1} \end{pmatrix}} AX} \to 0$$

$$t \longmapsto A^{-1}\begin{pmatrix} t & & & & \\ & t & & & \\ & & \ddots & & \\ & & & t & \\ & & & & 1/t^{n-1} \end{pmatrix} A$$

Examples

$$\begin{pmatrix} \epsilon^2 & & \\ & 1/\epsilon & \\ & & 1/\epsilon \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \epsilon^2 & & \\ & 1/\epsilon & \\ & & 1/\epsilon \end{pmatrix}$$

$$t \longmapsto \left(\begin{pmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{pmatrix}, \begin{pmatrix} t^2 & & \\ & t^{-1} & \\ & & t^{-1} \end{pmatrix} \right)$$

Examples

Examples

- Group Actions on Vector Spaces, Orbits & Orbit Closures
- Geometric Questions
- Conclusion

Conclusion

- Today we learned the basics about the geometric side of invariant theory
- Many examples of important group actions and their geometric problems
- Connections to other areas of mathematics
- Fundamental problems and theorems in geometric invariant theory
- Semistability theorem of Hilbert and Mumford