Lecture 10: Complexity of Ideal Membership Problem

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Overview

- Ideal Membership Problem & a Variant
- Univariate Case
- Multivariate Case
- EXPSPACE-completeness
- Conclusion
- Acknowledgements

- Input: $g_1, \ldots, g_s, f \in \mathbb{F}[x_1, \ldots, x_n]$ deg(g;), deg(f) $\leq d$
- **Output:** is $f \in (g_1, ..., g_s)$?

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$$f = g_1 \cdot h_1 + \dots + g_s h_s$$

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- But today we will see a different algorithm for it we will solve it by converting the polynomial system above into a *linear system* of equations

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- We know that if such polynomials exist then Groebner bases and the division algorithm will find them for us
- But today we will see a different algorithm for it we will solve it by converting the polynomial system above into a *linear system* of equations
- The complexity of today's algorithm comes from showing that if the *h_i*'s exist, then they must exist in some "reasonable degree"
- So we need to upper bound the degree of the h_i 's e_{x_i}

Algorithm - Main Idea

• If we know upper bound on the degree of the *h_i*'s then all we have left is a linear system!

$$f = g_{i}h_{i} + \cdots + g_{k}h_{k} \quad (*)$$

$$deg(l), deg(g_{i}) \leq d \quad deg(h_{i}) \leq D$$

$$\overline{\alpha} \in \{0, j, \dots, D\}^{n} \quad \overline{\alpha}^{\overline{\alpha}}$$

$$f_{\overline{\alpha}} = \sum_{i=1}^{n} \sum_{\overline{\beta} \leq \overline{\alpha}} \frac{g_{i}\overline{\beta}}{h_{p}\overline{\alpha}} + \frac{h_{i}\overline{\alpha}\overline{\beta}}{u_{n}h_{p}} \int_{\overline{\alpha}}^{c} coefficient of \overline{\alpha} \quad (*)$$

$$g_{lves} us D^{n} = quetions (linear)$$

Algorithm - Main Idea

- If we know upper bound on the degree of the h_i 's then all we have left is a linear system!
- Since linear systems can be solved in *polylogarithmic space*, a degree bound of <u>D</u> on the h_i's, together with a degree bound of <u>d</u> for <u>f</u>, g; would give us a space complexity of:

 $poly(n \log(D), \log(s))$

- Input: g_{ij} , $f_i \in \mathbb{F}[x_1, \ldots, x_n]$ where $i \in [s], j \in [t]$, $\deg(g_{ij}), \deg(f_i) \leq d$
- **Output:** is there h_1, \ldots, h_t such that

 $f_{i} = g_{i1}h_{1} + \dots + g_{it}h_{t} \quad \forall i \in [s]$ $G = (g_{ij}) \in [f [k]]^{k \times t}$ $G \begin{pmatrix} h_{i} \\ \vdots \\ h_{t} \end{pmatrix} = \begin{pmatrix} l_{i} \\ l_{z} \\ \vdots \\ l_{z} \end{pmatrix}$

- Input: $g_{ij}, f_i \in \mathbb{F}[x_1, \dots, x_n]$ where $i \in [s], j \in [t]$, $\deg(g_{ij}), \deg(f_i) \leq d$
- **Output:** is there h_1, \ldots, h_t such that

$$f_i = g_{i1}h_1 + \dots + g_{it}h_t \quad \forall i \in [s]$$

• Can be reduced to ideal membership problem by adding extra variables y_1, \ldots, y_s :

$$\underbrace{f_1y_1 + \dots + f_sy_s \in (y_1 \cdot g_{1j} + y_2 \cdot g_{2j} + \dots + y_s \cdot g_{sj})_{j=1}^t}_{h_s}$$

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 It will be convenient to prove that this problem can be solved in EXPSPACE

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Theorem (Hermann, Mayr-Meyer)

If the linear system of polynomials problem has a solution, then it has a solution in which

 $\deg(h_i) \leq (t \cdot d)^{2^n}$



• The above theorem proves that we can solve the ideal membership problem in EXPSPACE

Remarks

- The above theorem proves that we can solve the ideal membership problem in EXPSPACE
- We can assume that our base field $\mathbb F$ is infinite, without loss of generality.
- This is because a system of linear equations has a solution over an extension field $\mathbb{F}\subset\mathbb{K}$ if, and only if, it has a solution in \mathbb{F}
- Practice problem: prove this statement

• Ideal Membership Problem & a Variant

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- Rearranging columns, can write

$$M = \begin{pmatrix} A & v_1 & v_2 & \cdots & v_r \end{pmatrix}$$

where $A \in \mathbb{F}[x]^{s \times s}$ is invertible and r = t - sremaining columns

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$$M(h) = (f)$$

$$\overline{A(\binom{h_{1}}{i})} = (f) - \sum_{i=1}^{K} \frac{h_{sti} \cdot \mathcal{D}_{i}}{1}$$
for any choice of the formula over the solution over the so

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• By Cramer's rule $A^{-1} = \frac{\operatorname{Adj}(A)}{\det(A)} \xrightarrow{f \in A^{d}} \operatorname{Adj}(A) \xrightarrow{i \in A^{d}} \operatorname{Adj}(A) \xrightarrow{i \in A^{d}} \operatorname{Adj}(A)$

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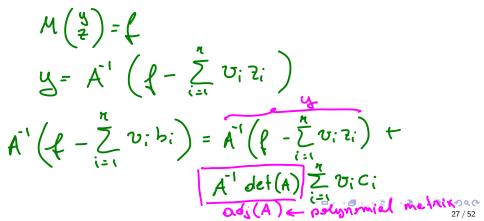
• By Cramer's rule $A^{-1} = \frac{\operatorname{Adj}(A)}{\operatorname{det}(A)}$

Ratio of polynomials of low degree!

• If h = (y, z) is a *polynomial* solution to Mh = f, then for any $c_1, \ldots, c_r \in \mathbb{F}[x]$ we have that $b_i = z_i - c_i \cdot \det(A)$ and

$$a = A^{-1}(f - b_1v_1 - \cdots - b_rv_r) = y + \operatorname{Adj}(A) \cdot (c_1v_1 + \cdots + c_rv_r)$$

gives another polynomial solution to $M(a, b)^T = f$.



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 Because we are in univariate case (thus we have Euclidean domain) we can assume that all z_i's are reduced modulo det(A) and thus have degree bounded by < l := deg(A) ≤ sd

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- Thus, we have

$$\frac{\deg(y) \leq \deg(A^{-1}) + \deg(f - z_1v_1 - \dots - z_rv_r)}{= \deg(\operatorname{Adj}(A)) - \deg(\det(A)) + \max\left\{ \deg(f), \deg(\sum_{i=1}^r z_iv_i) \right\}}$$

$$\leq (s-1)d - \ell + \max(\underline{d}, \ell - 1 + d) < sd \leq td$$

$$\deg(g) \leq td \qquad \deg(z) \leq td$$

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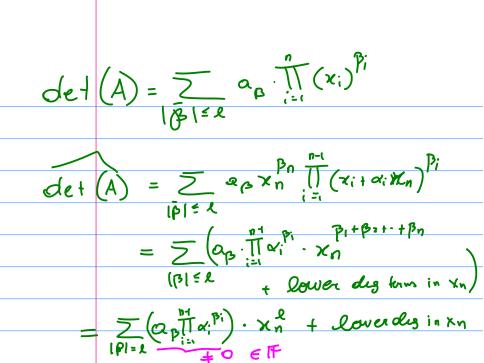
R[x] not Euclidean domain f(x) = (x^d + lower order terms) g(x) + 91(x) unit coefficient oleg (n) < d

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- Since $det(A) \neq 0$, a generic linear map as above will make

$$det(A) = \alpha x_n^{\ell} + \text{ (other terms of } x_n \text{ degree } < \ell)$$

$$det(A) = \sum_{\sigma \in S_n} (-1)^{\sigma} \cdot \prod_{i=1}^{\ell} \alpha_{i\sigma(i)}(x_{1,\cdots,x_n}) = \sum_{\sigma \in S_n} \alpha_{\overline{i}} \cdot \overline{x_i}$$



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$$f_{im}x_n^m = H_m^{(n)}[\underline{g_{i1}h_1} + \dots + \underline{g_{it}h_t}] \quad \forall i \in [s], m \in [td+d]$$

$$f_i = g_{i_1}h_{i_1}t \cdot t g_{i_k}h_{k_l}$$

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hom. components of deg m (voriable x_n)

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Thus our recursion becomes

$$\underline{D}(\underline{n},\underline{d},\underline{t}) \leq D(\underline{n-1},\underline{d},\underline{t^2d}) + td = D(n-1,d,(\underline{td})^2/\underline{d}) + \underline{td}$$

Recursion $\mathcal{D}(n, ol, t) \leq \mathcal{D}(n-1, d, (td)^2) + tol$ $\leq D(n-2, d, \left(\frac{(t_d)^2}{d}\right)^2 d) + \frac{(t_d)^2}{d} + \frac{(t_d)^2}{d}$ $= \mathcal{D}(n-2, d, \frac{(td)^{2}}{d}) + (td)^{2} + (td)$ $\leq \mathcal{D}(n-k, \sigma, (\frac{4d}{d})^{2^k}) + (\frac{2}{4d})^{2^{k-1}} + \cdots + (\frac{2}{4d})^{2^{k-1}}$ \Rightarrow (td)^{2ⁿ} M 지다는 지금은 지원은 지원이다. ж. 500 43 / 52

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- Setup: finite alphabet $\Sigma = \{\sigma_1, \dots, \sigma_r\}$, set of rewriting rules S (of the form $\alpha = \beta$ where $\alpha, \beta \in \Sigma^*$) where S contains the rules $\sigma_i \sigma_j = \sigma_j \sigma_i$

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- Input: two words $\alpha, \beta \in \Sigma^*$
- **Output:** is $\alpha = \beta$?

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- Mayr and Meyer also proved that the ideal membership problem is EXPSPACE-complete
- Reduced from the *commutative semigroup problem* (which they prove to be EXPSPACE hard) to ideal membership problem
- Setup: finite alphabet $\Sigma = \{\sigma_1, \dots, \sigma_r\}$, set of rewriting rules S (of the form $\alpha = \beta$ where $\alpha, \beta \in \Sigma^*$) where S contains the rules $\sigma_i \sigma_j = \sigma_j \sigma_i$ $\chi^{\prec} \times^{\mathfrak{P}} \in (\mathfrak{S})$ $\chi = \mathfrak{S} \in \mathfrak{S}$

 $\overline{x}^{\delta} - \overline{x}^{\delta}$

- Input: two words $\alpha, \beta \in \Sigma^*$
- **Output:** is $\alpha = \beta$?
- To reduce to ideal membership problem, need to rewrite the rules of S with polynomials, which they write as polynomials of the form $x^{\alpha} x^{\beta}$, then need to encode all these "relation polynomials" into a small ideal $\mathfrak{F} = (\mathfrak{F}_{1,2}, \mathfrak{F}_{n}) = \mathfrak{F}_{1,2}^{\mathfrak{F}_{n}} = \mathfrak{F}_{1,2}^{\mathfrak{F}_{n}}$

- Ideal Membership Problem & a Variant
- Univariate Case
- Multivariate Case
- EXPSPACE-completeness
- Conclusion
- Acknowledgements

Conclusion

- Different algorithm for Ideal Membership Problem and its analysis
- Reduced it to linear system solving!
- Saw degree bounds for the Ideal Membership Problem
- Would be interesting to see an analysis of the Groebner basis algorithm in case anyone wants to learn and teach it
- · EXPSPACE levilley
- what are natural problems encoding other complexity classes? - Can we have a first-grained complexity

Acknowledgement

• Lecture based entirely on the book by CLO: Ideals, varieties and algorithms (see course webpage for a copy - or get online version through UW library)