Re-demonstration of BS'83 thm on taking partial derivatives

BS'83: if $f \in \mathbb{F}[x_1,\dots,x_n]$ can computed by ckt Φ of size $S(\Phi) = s$ then there is a ckt Δ_Φ of size $O(s)$ s.t. Δ_Φ computes $\partial_1 f, \partial_2 f, \dots, \partial_n f$

Proof: induction on size of ckt

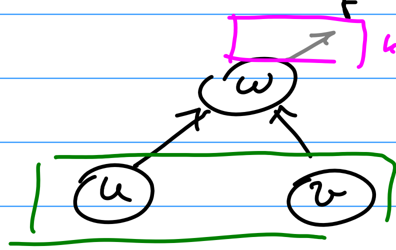
Base case: f input x_i or $\alpha \in \mathbb{F}$

Φ input (x_i) or (α) $\qquad \Delta_\Phi$ ckt $\boxed{(1) \quad (0)}$ Φ

$\partial_j x_i = \delta_{ij}$ $\qquad$ multi-output ckt.

Induction hypothesis: suppose claim is true for any ckt ψ s.t. $S(\psi) \le s-1$ $\qquad S(\Delta_\psi) = O(s-1)$

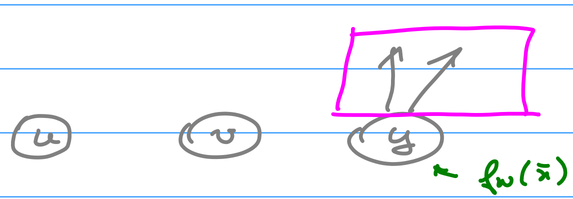
Φ take a bottom-most gate w which is not a leaf



$f_w = \begin{cases} \alpha u + \beta v \\ u \cdot v \end{cases}$

u, v inputs

$\partial_i f_w$ non-zero only constantly many values



$\Phi_{w=y}$

$S(\Phi_{w=y}) < S(\Phi)$

By induction hypothesis there is $\psi := \Delta_{\Phi_{w=y}}$

$S(\psi) = O(s-1)$

Δ_{Φ}

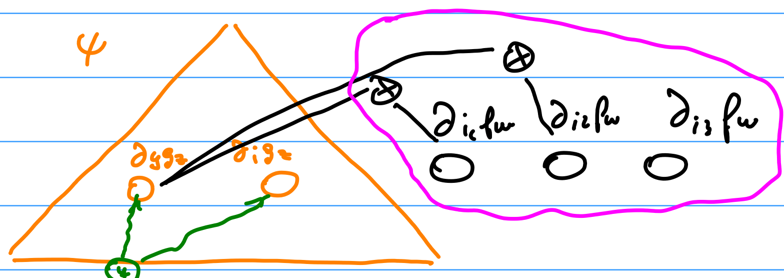
let $z \in \Phi$ $f_z(\bar{x})$ is poly computed by gate z in Φ let $g_z(\bar{x}, y)$ poly computed by gate z in $\Phi_{w=y}$ $\quad g_z(\bar{x}, f_w(\bar{x})) = f_z(\bar{x})$

Chain rule :

$$\partial_i f_z(\bar{x}) = \overbrace{\partial_i g_z(\bar{x}, y)}^{\text{in } \Psi \text{ it is derivative of gate in } \Phi_{w=y}} \Big|_{\underbrace{y = f_w}_{\text{constant number of wires to substitute}}} +$$

only new gates we need to add

$$+ \underbrace{\partial_i f_w}_{\text{this can only take constantly many values}} \cdot \underbrace{\partial_y g_z(\bar{x}, y)}_{\text{in } \Psi \text{ it is derivative of gate in } \Phi_{w=y}} \Big|_{y = f_w}$$



can compute all partial derivatives $\partial_i f_z$

Δf_w (w is in the bottom so constant size)

setting z to be output gate of Φ we conclude induction. $\blacksquare$