Lecture 3: Evaluation, Interpolation and Multiplication of Polynomials

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Overview

- Polynomial Evaluation
- Polynomial Multiplication
- Polynomial Interpolation
- Conclusion
- Acknowledgements
Polynomial Evaluation

Polynomial Multiplication

Polynomial Interpolation

Conclusion

Acknowledgements
Polynomial Evaluation

- **Setting:** ring $R[x]$, can perform basic operations ($+$, $\times$) over $R$
- **Input:** elements $\alpha, a_0, \ldots, a_d \in R$
- **Output:** $p(\alpha)$, where

\[ p(x) = a_0 + a_1x + \cdots + a_dx^d \in R[x] \]
Polynomial Evaluation

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- Naive algorithm:
  - Compute $\alpha^2, \alpha^3, \ldots, \alpha^d$  \hspace{1cm} ($d - 1$ multiplications)
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  - Compute $\alpha^2, \alpha^3, \ldots, \alpha^d$ $(d - 1$ multiplications$)$
  - Compute $a_j\alpha^j$ $(d$ multiplications$)$
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- **Naive algorithm:**
  - Compute $\alpha^2, \alpha^3, \ldots, \alpha^d$ (\(d - 1\) multiplications)
  - Compute $a_j\alpha^j$ (\(d\) multiplications)
  - Add $a_j\alpha^j$ (\(d\) additions)
  - Can we do better?

Horner's algorithm (a.k.a. Horner's rule):

Write $p(x) = \left( \ldots \left( (a_d x^d + a_{d-1})x + \cdots \right)x + a_1 \right)x + a_0$ with \(n\) multiplications and \(n\) additions
Polynomial Evaluation

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- **Can we do better?**
- **Horner’s algorithm (a.k.a. Horner’s rule):**
  - Write

\[
p(x) = \ldots ((a_dx + a_{d-1}) \cdot x + a_{d-2}) \cdot x + a_{d-3}) \cdot x + \cdots a_1) \cdot x + a_0
\]

\[
p(x) = 3x^2 - 2x + 1 = (3x - 2)x + 1
\]
Polynomial Evaluation

- **Setting:** ring $R[x]$, can perform basic operations ($\cdot$, $\times$) over $R$
- **Input:** elements $\alpha, a_0, \ldots, a_d \in R$
- **Output:** $p(\alpha)$, where

  $$p(x) = a_0 + a_1x + \cdots + a_dx^d \in R[x]$$

Naive algorithm:
- Compute $\alpha^2, \alpha^3, \ldots, \alpha^d$ ($2d-1$ multiplications)
- Compute $a_j\alpha^j$ ($d$ multiplications)
- Add $a_j\alpha^j$ ($d$ additions)

Can we do better?

Horner's algorithm (a.k.a. Horner's rule):
- Write

  $$p(x) = (\cdots ((a_dx + a_{d-1}) \cdot x + a_{d-2}) \cdot x + a_{d-3}) \cdot x + \cdots a_1) \cdot x + a_0$$

- $d$ multiplications and $d$ additions
Polynomial Evaluation

- **Setting:** ring $\mathbb{R}[x]$, can perform basic operations ($+$, $\times$) over $\mathbb{R}$
- **Input:** elements $\alpha, a_0, \ldots, a_d \in \mathbb{R}$
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$$p(x) = \left(\cdots\left(\left(a_dx + a_{d-1}\right)\cdot x + a_{d-2}\right)\cdot x + a_{d-3}\right)\cdot x + \cdots a_1\right)\cdot x + a_0$$

  - $n$ multiplications and $n$ additions

- **Ostrowski’1954:** Is Horner’s rule optimal for polynomial evaluation?
Different Cost Function

- From previous lecture, multiplying integers may be harder than adding integers (same problem for matrix rings)
- **Open problem**: is integer addition easier than integer multiplication?

See resources and final project page of the course to find exciting new developments on this question! In [Harvey, van der Hoeven 2019] the authors find an $O(n \log n)$ algorithm for multiplying two integers!
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- Ostrowski’s *non-scalar complexity*
  
  \( \mathbb{F} \) is a field, \( R = \mathbb{F}[\alpha, a_0, \ldots, a_d] \)
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- Ostrowski’s *non-scalar complexity*
  1. $\mathbb{F}$ is a field, $R = \mathbb{F}[\alpha, a_0, \ldots, a_d]$
  2. **Scalar operations:** addition of two elements from $R$, multiplication of element from $R$ by fixed constant from $\mathbb{F}$ (fixed by algorithm).
  3. **Non-scalar operations:** all other operations
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- Can one improve the non-scalar operations in polynomial evaluation?
- [Pan 1966] No! Horner’s rule is optimal! \[ \text{lower bound} \]

$$\text{input: } \alpha, a_0, \ldots, a_d$$
Evaluating a fixed polynomial

- Not all polynomials are created equal.
- What if we want to evaluate a particular polynomial? Say we know coefficients $a_0, a_1, \ldots, a_d \in \mathbb{F}$ and

$$p(x) = a_0 + a_1 x + \cdots + a_d x^d \in \mathbb{F}[x]$$

$a_0, a_1, \ldots, a_d$ are scalars
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- **Input:** value $\alpha \in \mathbb{F}$
- **Output:** evaluation $p(\alpha)$
- [Paterson, Stockmeyer 1973]: $p(\alpha)$ can be evaluated with $2\lceil \sqrt{d} \rceil - 1$ non-scalar multiplications.
  - partition $p$ into $\sqrt{d}$ blocks of length $\sqrt{d}$. Say $m = \lceil \sqrt{d} \rceil$ and $k = \lceil d/m \rceil + 1$.
  \[
  p(x) = (a_{km-1}x^{m-1} + \cdots + a_{(k-1)m}) \cdot x^{(k-1)m} + \cdots
  \]
  \[
  \cdots + (a_{2m-1}x^{m-1} + \cdots + a_m) \cdot x^m + (a_{m-1}x^{m-1} + \cdots + a_0)
  \]

- $km - 1 > d \implies a_{km-1} = 0$
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  - Compute $\alpha, \alpha^2, \ldots, \alpha^m$ using $m - 1$ non-scalar operations
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  - Compute $\alpha, \alpha^2, \ldots, \alpha^m$ using $m - 1$ non-scalar operations
  - Compute $\beta_j = a_{jm-1}\alpha^{m-1} + \cdots + a_{(j-1)m}$ no cost (scalar ops)
Evaluating a fixed polynomial

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- Compute $\alpha, \alpha^2, \ldots, \alpha^m$ using $m - 1$ non-scalar operations
- Compute $\beta_j = a_{km-1}\alpha^{m-1} + \cdots + a_{(k-1)m}$ no cost (scalar ops)
- Horner's rule on $\sum_{j=0}^k \beta_j \cdot \alpha^{jm}$ \hspace{1cm} $k - 1$ non-scalar

---

Baby-steps, giant-steps evaluation.
Evaluating a fixed polynomial

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  - Baby-steps, giant-steps evaluation.
Polynomial Evaluation

Polynomial Multiplication

Polynomial Interpolation

Conclusion

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Polynomial Multiplication

- In lecture 1 we saw how to multiply two polynomials of degree $d$ in time $O(d^2)$ (computational model ≠ ring operations)
- Can we do better?
Polynomial Multiplication

In lecture 1 we saw how to multiply two polynomials of degree $d$ in time $O(d^2)$ (computational model # ring operations)

Can we do better?

YES. Assume $d = 2^k$, and $P, Q \in R[x]$ are of degree $< d$. Let $m = d/2$. 

Rewrite:

$P(x) = P_1(x) \cdot x^m + P_0(x)$

$Q(x) = Q_1(x) \cdot x^m + Q_0(x)$

Now $P(x) \cdot Q(x) = P_1(x)Q_1(x) x^d + (P_1(x)Q_0(x) + P_0(x)Q_1(x)) x^m + P_0(x)Q_0(x)$

Reduce multiplication of two polynomials of degree $< d$ to 4 multiplications of polynomials of degree $< d/2$.

Following master's theorem, this does not help us...
Polynomial Multiplication

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- YES. Assume \(d = 2^k\), and \(P, Q \in R[x]\) are of degree \(<d\). Let \(m = d/2\).
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\[
P(x) = P_1(x) \cdot x^m + P_0 \\
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\]

\(P_0, P_1, Q_0, Q_1\) polynomials of degree \(<m = \frac{d}{2}\)
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- Rewrite:

$$P(x) = P_1(x) \cdot x^m + P_0 \quad Q(x) = Q_1(x) \cdot x^m + Q_0(x)$$

- Now

$$P(x) \cdot Q(x) = P_1 Q_1 x^d + (P_1 Q_0 + P_0 Q_1) x^m + P_0 Q_0$$
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\]

- Reduce multiplication of two polynomials of degree \(< d \) to 4 multiplications of polynomials of degree \(< d/2 \)

\[
T(d) \leq 4 \cdot T(d/2) + O(d) \Rightarrow O(d^2)
\]
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Reduce multiplication of two polynomials of degree < \( d \) to 4 multiplications of polynomials of degree < \( d/2 \)

Following master’s theorem, this does not help us...
Karatsuba & Ofman’s trick (1965)

- Can we reduce number of multiplications (perhaps at the cost of doing more additions)?

\[ PQ = P_1 Q_1 (x^d - x^m) + (P_1 + P_0)(Q_1 + Q_0)x^m + P_0 Q_0 (1 - x^m) \]

Remark: multiplication by power of \( x \) doesn’t count as multiplication, as this only shifts the coefficients of the polynomial.

Now we have reduced multiplication of two polynomials of degree \( <d \) to 3 multiplications of polynomials of degree \( <d/2 \).

By master’s theorem, we get that Karatsuba-Ofman method can be done with \( O(d \log_2 3) = O(d^{1.59}) \) ring operations.
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\[ PQ = P_1Q_1x^d + (P_1Q_0 + P_0Q_1)x^m + P_0Q_0 (P_1 + P_0)(Q_1 + Q_0) \]

Extra form: \[ P_1Q_1x^m + P_0Q_0x^m \]
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- By master’s theorem, we get that Karatsuba-Ofman method can be done with \( O(d^{\log_2 3}) = O(d^{1.59}) \) ring operations.

previously \( O(d^2) \)
Complexity of Karatsuba-Ofman

- If $T(2^k) \leq 3T(2^{k-1}) + c \cdot 2^k$ then $T(2^k) \leq 3^k - 2c \cdot 2^k$ for $k \geq 1$.

$T(d)$ time to multiply 2 poly deg < d

Proof in by induction.
Complexity of Karatsuba-Ofman

- If \( T(2^k) \leq T(2^{k-1}) + c \cdot 2^k \) then \( T(2^k) \leq 3^k - 2c \cdot 2^k \) for \( k \geq 1 \).

\[
3\log_2 d = 2\log_2 3 \cdot \log_2 d = d^{\log_2 3}
\]

\[
T(2^k) \leq 3^k - 2c \cdot 2^k
\]

\[
k = \log_2 d \quad 2^k = d
\]

\[
T(d) \leq 3^{\log_2 d} = 2^{\log_3 \log d} = d^{\log_3}
\]
- Polynomial Evaluation
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Polynomial Interpolation

- **Problem:** given $d + 1$ evaluations of a polynomial $p(x) \in F[x]$ of degree $\leq d$, can we “reconstruct” the polynomial $p$ (as list of coefficients)?

- **Input:** evaluations $p(u_0), \ldots, p(u_d)$ of polynomial $p(x)$ of degree $\leq d$

- **Output:** coefficients $(p_0, p_1, \ldots, p_d)$ of $p(x)$

$$p(x) = p_0 + p_1x + \cdots + p_dx^d$$

Closely related to matrix-vector multiplication:

$$
\begin{bmatrix}
  u_0 & 0 & u_1 & 0 & \cdots & u_d & 0 \\
  u_0 & 1 & u_1 & 1 & \cdots & u_d & 1 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  u_0 & d & u_1 & d & \cdots & u_d & d \\
\end{bmatrix}
\begin{bmatrix}
  p_0 \\
  p_1 \\
  \vdots \\
  p_d \\
\end{bmatrix}
= 
\begin{bmatrix}
  p(u_0) \\
  p(u_1) \\
  \vdots \\
  p(u_d) \\
\end{bmatrix}
$$

Interpolation amounts to inverting Vandermonde matrix!
Polynomial Interpolation

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- **Input:** evaluations \( p(u_0), \ldots, p(u_d) \) of polynomial \( p(x) \) of degree \( \leq d \)
- **Output:** coefficients \((p_0, p_1, \ldots, p_d)\) of \( p(x) \)

Closely related to matrix-vector multiplication:

\[
\begin{pmatrix}
    u_0^0 & u_0^1 & \cdots & u_0^d \\
    u_1^0 & u_1^1 & \cdots & u_1^d \\
    \vdots & \vdots & \ddots & \vdots \\
    u_d^0 & u_d^1 & \cdots & u_d^d \\
\end{pmatrix}
\begin{pmatrix}
    p_0 \\
    p_1 \\
    \vdots \\
    p_d \\
\end{pmatrix}
= 
\begin{pmatrix}
    p(u_0) \\
    p(u_1) \\
    \vdots \\
    p(u_d) \\
\end{pmatrix}
\]

Interpolation amounts to inverting Vandermonde matrix!

Will use this idea later in course to obtain faster algorithm for polynomial multiplication.
Polynomial Interpolation

- **Problem:** given $d + 1$ evaluations of a polynomial $p(x) \in \mathbb{F}[x]$ of degree $\leq d$, can we “reconstruct” the polynomial $p$ (as list of coefficients)?

- **Input:** evaluations $p(u_0), \ldots, p(u_d)$ of polynomial $p(x)$ of degree $\leq d$

- **Output:** coefficients $(p_0, p_1, \ldots, p_d)$ of $p(x)$

Closely related to matrix-vector multiplication:

\[
\begin{pmatrix}
  u_0^0 & u_0^1 & \cdots & u_0^d \\
  u_1^0 & u_1^1 & \cdots & u_1^d \\
  \vdots & \vdots & \ddots & \vdots \\
  u_d^0 & u_d^1 & \cdots & u_d^d \\
\end{pmatrix}
\begin{pmatrix}
  p_0 \\
  p_1 \\
  \vdots \\
  p_d \\
\end{pmatrix}
= 
\begin{pmatrix}
  p(u_0) \\
  p(u_1) \\
  \vdots \\
  p(u_d) \\
\end{pmatrix}
\]

Interpolation amounts to inverting Vandermonde matrix!

\[
V \begin{pmatrix}
  p_0 \\
  p_1 \\
  \vdots \\
  p_d \\
\end{pmatrix} = \begin{pmatrix}
  p(u_0) \\
  p(u_1) \\
  \vdots \\
  p(u_d) \\
\end{pmatrix} \implies \begin{pmatrix}
  p_0 \\
  p_1 \\
  \vdots \\
  p_d \\
\end{pmatrix} = V^{-1} \begin{pmatrix}
  p(u_0) \\
  p(u_1) \\
  \vdots \\
  p(u_d) \\
\end{pmatrix}
\]
Polynomial Interpolation

- **Problem:** given $d + 1$ evaluations of a polynomial $p(x) \in \mathbb{F}[x]$ of degree $\leq d$, can we “reconstruct” the polynomial $p$ (as list of coefficients)?

- **Input:** evaluations $p(u_0), \ldots, p(u_d)$ of polynomial $p(x)$ of degree $\leq d$

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- Closely related to matrix-vector multiplication:

$$
\begin{pmatrix}
    u_0^0 & u_1^0 & \cdots & u_d^0 \\
    u_1^0 & u_1^1 & \cdots & u_1^d \\
    \vdots & \vdots & \ddots & \vdots \\
    u_d^0 & u_d^1 & \cdots & u_d^d \\
\end{pmatrix}
\begin{pmatrix}
    p_0 \\
    p_1 \\
    \vdots \\
    p_d \\
\end{pmatrix}
= 
\begin{pmatrix}
    p(u_0) \\
    p(u_1) \\
    \vdots \\
    p(u_d) \\
\end{pmatrix}
$$

- Interpolation amounts to inverting Vandermonde matrix!

- Will use this idea later in course to obtain faster algorithm for polynomial multiplication
Polynomial Interpolation
Theorem

Let $\mathbb{F}$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $\mathbb{F}[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

Karatsuba-Ofman $O(d^{\frac{5}{3}})$ operations

($\times$ing operations)
Let $\mathbb{F}$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $\mathbb{F}[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

- Pick $2d + 1$ distinct scalars $u_0, \ldots, u_{2d} \in \mathbb{F}$

(hardcode these values in our machine, optimize operations with them)
Polynomial Multiplication Non-Scalar Setting

**Theorem**

Let $\mathbb{F}$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $\mathbb{F}[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

- Pick $2d + 1$ distinct scalars $u_0, \ldots, u_{2d} \in \mathbb{F}$
- Evaluate $p(u_i), q(u_i)$. (no cost - only scalar multiplications)

\[
p(x) = p_0 + p_1 x + \ldots + p_d x^d
\]

$p_i$ scalars ($\in \mathbb{F}$)

$u_i$ scalars

$u_i^k$ scalar

$pu_i$ scalar
**Theorem**

Let $\mathbb{F}$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $\mathbb{F}[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

- Pick $2d + 1$ distinct scalars $u_0, \ldots, u_{2d} \in \mathbb{F}$
- Evaluate $p(u_i), q(u_i)$. (no cost - only scalar multiplications)
- Compute $\gamma_i = p(u_i)q(u_i)$ (2d + 1 non-scalar multiplications)

$$p(u_i) \cdot q(u_i) = (p \cdot q)(u_i)$$

Ostrowski's model: non-scalar operations

2d+1 evaluations of $(p \cdot q)(x)$

$$\text{degree } \leq 2d$$
Theorem

Let $\mathbb{F}$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $\mathbb{F}[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

- Pick $2d + 1$ distinct scalars $u_0, \ldots, u_{2d} \in \mathbb{F}$
- Evaluate $p(u_i), q(u_i)$. (no cost - only scalar multiplications)
- Compute $\gamma_i = p(u_i)q(u_i)$ (2d + 1 non-scalar multiplications)
- Lagrange polynomial:
  \[ L_i(x) = \prod_{j \neq i} \frac{x - u_j}{u_i - u_j} \]

\[ L_i(u_j) = \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } i = j \end{cases} \]
Polynomial Multiplication Non-Scalar Setting

**Theorem**

Let $F$ be a field with $\geq 2d + 1$ elements. Polynomial multiplication over $F[x]$ in the non-scalar model of two polynomials of degree $\leq d$ can be done with $2d + 1$ non-scalar multiplications.

- Pick $2d + 1$ distinct scalars $u_0, \ldots, u_{2d} \in F$
- Evaluate $p(u_i), q(u_i)$. (no cost - only scalar multiplications)
- Compute $\gamma_i = p(u_i)q(u_i)$ (2d + 1 non-scalar multiplications)
- Lagrange polynomial:

$$L_i(x) = \prod_{j \neq i} \frac{x - u_j}{u_i - u_j}$$

- $p(x)q(x) = \sum_{i=0}^{2d} \gamma_i \cdot L_i(x)$

- only using scalar multiplications
Polynomial multiplication

\[ p(x) \cdot q(x) = \sum_{i=0}^{2d} \delta_i L_i(x) \quad \delta_i = p(u_i)q(u_i) \]

\[ p(u_j) \cdot q(u_j) = \sum_{i=0}^{2d} \delta_i L_i(u_j) = \delta_j \cdot 1 \]

\[ \{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{array} \]

Two expansions of polynomials
\[ \begin{cases} \begin{align*} p(x) \cdot q(x) & = \sum_{i=0}^{2d} \delta_i L_i(x) \\ \delta_i & = p(u_i)q(u_i) \end{align*} \end{cases} \]

of degree \( \leq 2d \) which agree on \( 2d+1 \) distinct evaluations \( \Rightarrow \) these two polynomials are the same!
• Polynomial Evaluation

• Polynomial Multiplication

• Polynomial Interpolation

• Conclusion

• Acknowledgements
In today’s lecture, we learned

- computational models for measuring complexity of multiplication, evaluation and interpolation
  - ring operations
  - non-scalar complexity
- Algorithms for
  - polynomial evaluation
  - polynomial multiplication
  - polynomial interpolation
Acknowledgement

Based largely on Arne’s notes

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