Lecture 24: Complexity of Ideal Membership Problem

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April 12, 2021

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Overview

- Ideal Membership Problem & a Variant
- Univariate Case
- Multivariate Case
- EXPSPACE-completeness
- Conclusion
- Acknowledgements

- Input: $g_1, \ldots, g_s, f \in \mathbb{F}[x_1, \ldots, x_n]$
- **Output:** is $f \in (g_1, ..., g_s)$?

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$$f = g_1 \cdot h_1 + \dots + g_s h_s$$

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- But today we will see a different algorithm for it we will solve it by converting the polynomial system above into a *linear system* of equations

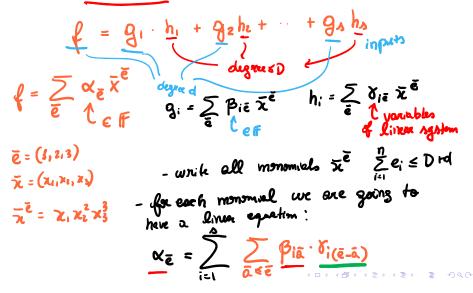
- Input: $g_1, \ldots, g_s, f \in \mathbb{F}[x_1, \ldots, x_n]$
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- We know that if such polynomials exist then Groebner bases and the division algorithm will find them for us
- But today we will see a different algorithm for it we will solve it by converting the polynomial system above into a *linear system* of equations
- The complexity of today's algorithm comes from showing that if the *h_i*'s exist, then they must exist in some "reasonable degree"
- So we need to upper bound the degree of the h_i 's

Algorithm - Main Idea Pia xa · Viller J

 If we know upper bound on the degree of the h_i's then all we have left is a linear system!

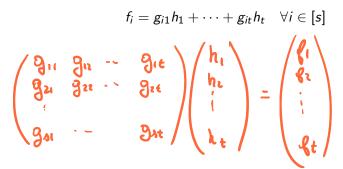


Algorithm - Main Idea

- If we know upper bound on the degree of the h_i 's then all we have left is a linear system!
- Since linear systems can be solved in *polylogarithmic space*, a degree bound of D on the h_i's, together with a degree bound of d for f_i, g would give us a space complexity of:

• Input: g_{ij} , $f_i \in \mathbb{F}[x_1, \dots, x_n]$ where $i \in [s], j \in [t]$, $\deg(g_{ij}), \deg(f_i) \leq d$

• **Output:** is there h_1, \ldots, h_t such that



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$$f_i = g_{i1}h_1 + \cdots + g_{it}h_t \quad \forall i \in [s]$$

 Can be reduced to ideal membership problem by adding extra variables y₁,..., y_s:

$$f_1y_1 + \cdots + f_sy_s \in (y_1 \cdot g_{1j} + y_2 \cdot g_{2j} + \cdots + y_s \cdot g_{sj})_{j=1}^t$$

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 It will be convenient to prove that this problem can be solved in EXPSPACE

- Input: $g_{ij}, f_i \in \mathbb{F}[x_1, \dots, x_n]$ where $i \in [s], j \in [t]$, $\deg(g_{ij}), \deg(f_i) \leq d$
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Theorem (Hermann, Mayr-Meyer)

If the linear system of polynomials problem has a solution, then it has a solution in which

 $\deg(h_i) \leq (t \cdot d)^{2^n}$



• The above theorem proves that we can solve the ideal membership problem in EXPSPACE

Remarks

- The above theorem proves that we can solve the ideal membership problem in EXPSPACE
- We can assume that our base field 𝔽 is infinite, without loss of generality.
- This is because a system of linear equations has a solution over an extension field $\mathbb{F}\subset\mathbb{K}$ if, and only if, it has a solution in \mathbb{F}
- Practice problem: prove this statement

• Ideal Membership Problem & a Variant

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- If s = t then M is invertible and our solution would be $\mathbf{h} = M^{-1}\mathbf{f}$
- Rearranging columns, can write

$$M = \begin{pmatrix} A & v_1 & v_2 & \cdots & v_r \end{pmatrix}$$

where $A \in \mathbb{F}[x]^{s \times s}$ is invertible and $r = t - s$

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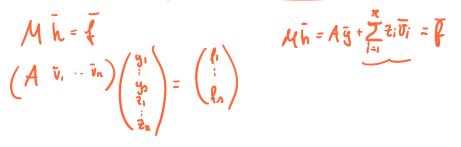
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$$\mathbf{y} = \mathcal{A}^{-1} \cdot \left(\mathbf{f} - \sum_{i=1}^r z_i \mathbf{v}_i
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• By Cramer's rule $A^{-1} = \frac{\operatorname{Adj}(A)}{\operatorname{det}(A)}$

Ratio of polynomials of low degree!

• If $\mathbf{h} = (\mathbf{y}, \mathbf{z})$ is a *polynomial* solution to $M\mathbf{h} = \mathbf{f}$, then for any $c_1, \ldots, c_r \in \mathbb{F}[x]$ we have that $b_i = z_i - c_i \cdot \det(A)$ and $\mathbf{a} = A^{-1}(\mathbf{f} - b_1\mathbf{v}_1 - \cdots - b_r\mathbf{v}_r) = \underline{y} + \operatorname{Adj}(A) \cdot (c_1\mathbf{v}_1 + \cdots + c_r\mathbf{v}_r)$ gives another polynomial solution to $M(\mathbf{a}, \mathbf{b})^T = \mathbf{f}$. from polynomial solution $\bar{h} = (\frac{5}{2})$ can construct another polynomial solution $b_i = 2i - c_i det(A)$ $\overline{\mathbf{Q}} = A^{-1} \left(\sqrt{\mathbf{I}} - b_1 \overline{\mathbf{v}}_1 - \cdots - b_n \overline{\mathbf{v}}_n \right) \qquad \sqrt{\frac{Adj V \mathbf{v}}{\mathcal{I} \mathbf{L}^+}}$ $= A^{-1} \left(\overline{\mathbf{I}} - 2_1 \overline{\mathbf{v}}_1 - \cdots - 2_n \overline{\mathbf{v}}_n \right) + A^{-1} \cdot det(A) \left(c_1 \overline{\mathbf{v}}_1 + \cdots + c_n \overline{\mathbf{v}}_n \right)$ ロト (日) (日) (日) (日) (日) (日)

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gives another polynomial solution to $M(\mathbf{a}, \mathbf{b})^T = \mathbf{f}$.

Because we are in univariate case (thus we have Euclidean domain) we can assume that all z_i's are reduced modulo det(A) and thus have degree bounded by < l := det(A) ≤ sd
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$$F[x] = C_i \det(A) + \underbrace{b_i}_{deg}(b_i) < \deg(\det(A))$$
Euclidean
$$A_{Nijoism}$$

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- Because we are in univariate case (thus we have Euclidean domain) we can assume that all z_i's are reduced modulo det(A) and thus have degree bounded by < ℓ := deg(A) ≤ sd
- Thus, we have

$$\begin{split} \deg(y) &\leq \deg(A^{-1}) + \deg(f - z_1 v_1 - \dots - z_r v_r) \\ &= \deg(\operatorname{Adj}(A)) - \deg(\det(A)) + \max\left\{ \operatorname{deg}(f), \operatorname{deg}(\sum_{i=1}^r z_i v_i) \right\} \\ &\leq (s-1)d - \ell + \max(d, \ell-1+d) < sd \leq td \quad (A \leq \epsilon) \\ \text{(such : if thus is a solution then is one with degree 3d} \end{split}$$

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- But Euclidean Divison still works if the polynomials are *monic* in x_n (so all we need is that det(A) be monic over x_n) det (A) monic in xn => we can control degree in Xn R[x] not necessarily Eaclidian domain f(x) = x^d + lower order ferns g(x) = a, x + $g(x) = q(x) \cdot f(x) + y(x)$ $a(x) - a_n$

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 Since det(A) ≠ 0, a generic linear map as above will make det(A) = αx_n^ℓ + (other terms of x_n degree < ℓ)
 det(A) (x₁,...,x_n) cet(A) (α₁x_n) α₂x_n × n × n ≠ 0

• As in the univariate case, and because we can make det(A) monic in x_n we can reduce to solutions where $deg_n(h)$ is upper bounded by $t \cdot d$

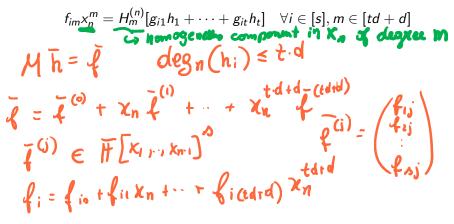
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$$f_{im}x_n^m = H_m^{(n)}[g_{i1}h_1 + \dots + g_{it}h_t] \quad \forall i \in [s], m \in [td+d]$$

• System above has s(t+1)d equations of polynomials in $\mathbb{F}[x_1, \ldots, x_{n-1}]$ of degree $\leq d$

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- And $\leq t \cdot td$ unknowns given by the coefficients

$$h_k = \sum_{i=0}^{td-1} (h_{ki}) x_n^i$$

new variable vector
td new vectors

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• Thus our recursion becomes

$$D(n, d, t) \leq D(n-1, d, t^{2}d) + td = D(n-1, d, (td)^{2}/d) + td$$
in parally degree of the second of the secon

Recursion

 $D(n,d,t) \in D(n-1,d,t^2d) + td$ $\leq \mathbb{D}\left(\frac{n-2}{d}, \frac{d}{d}, \frac{d}{d}\right)^{2} \cdot d + \frac{d}{d} + d$ $\leq D(n-k,d,\frac{(t_d)^{2^k}}{d}) + O(\frac{(t_d)^{2^k}}{d})$ $\leq (td)^{2} \leq (td)^{1}$

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• Setup: finite alphabet $\Sigma = \{\sigma_1, \dots, \sigma_r\}$, set of rewriting rules S (of the form $\alpha = \beta$ where $\alpha, \beta \in \Sigma^*$) where S contains the rules $\sigma_i \sigma_j = \sigma_j \sigma_i$ communicative matrix $\sigma_1 \sigma_1 \sigma_2 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \in \Sigma^*$ $\sigma_1 \sigma_2 \sigma_3 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 \sigma_3 \in \Sigma^*$

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- Input: two words $\alpha, \beta \in \Sigma^*$
- **Output:** is $\alpha = \beta$?

Commutative Semigroup problem

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- Mayr and Meyer also proved that the ideal membership problem is EXPSPACE-complete
- Reduced from the *commutative semigroup problem* (which they prove to be EXPSPACE hard) to ideal membership problem
- Setup: finite alphabet $\Sigma = \{\sigma_1, \dots, \sigma_r\}$, set of rewriting rules S (of the form $\alpha = \beta$ where $\alpha, \beta \in \Sigma^*$) where S contains the rules $\sigma_i \sigma_j = \sigma_j \sigma_i$
- Input: two words $\alpha, \beta \in \Sigma^*$
- **Output:** is $\alpha = \beta$?
- To reduce to ideal membership problem, need to rewrite the rules of S with polynomials, which they write as polynomials of the form $x^{\alpha} x^{\beta}$, then need to encode all these "relation polynomials" into a small ideal

Conclusion

- Different algorithm for Ideal Membership Problem and its analysis
- Reduced it to linear system solving!
- Saw degree bounds for the Ideal Membership Problem

Acknowledgement

• Lecture based entirely on Madhu's notes, lecture 14 http://people.csail.mit.edu/madhu/FT98/

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