Lecture 19: Matrix Multiplication & Fast Modular Composition

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Overview

- Fast Linear Algebra
- Matrix Multiplication
- Fast Modular Composition
- Conclusion

Importance of Fast Linear Algebra

- So far we have discussed how to get better algorithms for:
  1. integer & polynomial multiplication
  2. integer & polynomial division
  3. polynomial factoring
  4. polynomial interpolation
  5. integer & polynomial GCD
- Also saw algorithms for algebraic geometry problems and invariant theoretic problems

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  1. evaluation of determinant
  2. matrix multiplication
  3. linear system solving
  4. find rank, basis for null space, Jordan form, etc

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  1. evaluation of determinant
  2. matrix multiplication
  3. linear system solving
  4. find rank, basis for null space, Jordan form, etc
- These tasks are pervasive in symbolic computation (and in real life!)

Matrix Multiplication

square matrices

- **Input:** matrices $A, B \in \mathbb{F}^{n \times n}$
- **Output:** product $C = AB$

multiplying rectangular matrices $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times p}$

is also quite important in theory and in practice!

Matrix Multiplication

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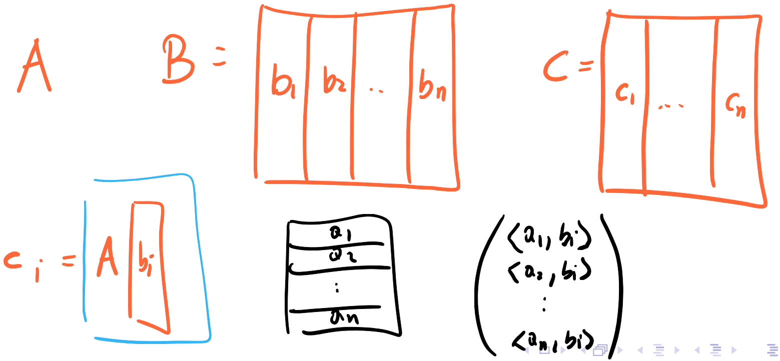
- Naive algorithm:

  Compute n matrix vector multiplications.



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  Can we do better?

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  Can we do better?
- Strassen 1969: YES!
- Idea: divide matrix into blocks, and *reduce number of multiplications* needed!

  Similar in spirit as Karatsuba's algorithm for polynomial multiplication!

Strassen's Algorithm

- Suppose that $n = 2^k$ (doesn't affect our asymptotic running time)
- Let $A, B, C \in \mathbb{F}^{n \times n}$ such that $C = AB$. Divide them into blocks of size $n/2$:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$C_{11} = A_{11}B_{11} + A_{12}B_{21}$

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$C_{21} = A_{21}B_{11} + A_{22}B_{21}$

$C_{22} = A_{21}B_{12} + A_{22}B_{22}$

8 multiplications

$T(n) \leq 8 \cdot T(n/2) + c \cdot \left(\frac{n}{2}\right)^2$

Master thm

$T(n) = O(n^3)$

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- Define following matrices:

$$S_1 = A_{21} + A_{22},\ S_2 = S_1 - A_{11},\ S_3 = A_{11} - A_{21},\ S_4 = A_{12} - S_2$$

$$T_1 = B_{12} - B_{11},\ T_2 = B_{22} - T_1,\ T_3 = B_{22} - B_{12},\ T_4 = T_2 - B_{21}$$

takes us $O((\frac{n}{2})^2)$ time to compute them.

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- Compute the following 7 products:

$$P_1 = A_{11}B_{11},\ P_2 = A_{12}B_{21},\ P_3 = S_4B_{22},\ P_4 = A_{22}T_4$$

$$P_5 = S_1T_1,\ P_6 = S_2T_2,\ P_7 = S_3T_3$$

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$$A_{11}B_{11} + (A_{12} + A_{11} - A_{21} - A_{22})B_{22} + (A_{21} + A_{22})(B_{12} - B_{11})$$
$$+ (A_{21} + A_{22} - A_{11})(B_{22} - B_{12} + B_{11})$$

$$A_{12}B_{22} + A_{11}B_{12}$$

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- Correctness follows from the computations

Analysis of Strassen's Algorithm

- To compute $AB = C$ we used:
  1. 8 additions — S_i, T_i's
  2. 7 multiplications — P_i's
  3. 10 additions — C_{ij}'s

Analysis of Strassen's Algorithm

- To compute $AB = C$ we used:
  1. 8 additions *set up multiplications* — S_i, T_i's
  2. 7 multiplications — P_i's
  3. 10 additions *put everything together* — C_{ij}'s
- Recurrence:

$$MM(n) \leq 7 \cdot MM(n/2) + 18 \cdot c \cdot (n/2)^2$$

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- Recurrence:

$k = \log_2 n$

$$MM(n) \leq 7 \cdot MM(n/2) + 18 \cdot c \cdot (n/2)^2$$

$$MM(2^k) \leq 7 \cdot MM(2^{k-1}) + 18 \cdot c \cdot 2^{2k-2}$$

$$MM(2^k) \leq 7^k \cdot 8 + 18 \cdot c \cdot \sum_{j=1}^{k-1} 4^j = O(7^k) = O(n^{\log_2 7})$$

$$\sum_{j=1}^{k-1} 4^j \to \frac{4^k - 1}{3}$$

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- Could also use Master theorem to get $MM(n) = O(n^{\log 7}) \approx O(n^{2.807})$

Matrix Multiplication Exponent

- We can define ω (or ω_{mult}) as the *matrix multiplication exponent*.
  1. If an algorithm for $n \times n$ matrix multiplication has running time $O(n^{\alpha})$, then $\omega \leq \alpha$.
  2. For any $\varepsilon > 0$, there is an algorithm for $n \times n$ matrix multiplication running in time $O(n^{\omega+\varepsilon})$

ω is the least (infimum) of all feasible exponents for matrix multiplication

We know: $\omega \geq 2$

Open question: is $\omega = 2$?

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- As we will see later in the course, ω is a fundamental constant in computer science!
- Currently we know $\omega < 2.376$

Open Question

What is the right value of ω?

Historical Remarks

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- Motivated work on better algorithms for all other linear algebraic problems
- introduced complexity of computation of *bilinear functions* and the study of complexity of tensor decompositions

- Fast Linear Algebra
- Matrix Multiplication
- Fast Modular Composition
- Conclusion

Modular Composition

- **Input:** polynomials $f, g, h \in \mathbb{F}[x]$ such that

$$\deg(g), \deg(h) < \deg(f) = n.$$

- **Output:** $g(h) \mod f$ remainder of composition $g(h)$ modulo f

$$g(x) = x^2 + 1 \qquad h(x) = x^3 + 1$$

$$g(h) = (x^3+1)^2 + 1 = x^6 + 2x^3 + 2$$

$$f = x^4 \qquad g(h) \equiv 2x^3 + 2 \mod f$$

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- If we only use Horner's rule, this can be done with $O(n \cdot M(n))$ operations in $\mathbb{F}$

$M(n) :=$ time it takes to multiply two polynomials of degree $\leq n$.

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- If we only use Horner's rule, this can be done with $O(n \cdot M(n))$ operations in $\mathbb{F}$
- We can do much better by combining: *fast polynomial arithmetic* and *fast matrix arithmetic*

Fast Algorithm

- **Input:** polynomials $f, g, h \in \mathbb{F}[x]$ such that

$$\deg(g), \deg(h) < \deg(f) = n = m^2$$

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- **Output:** $g(h) \mod f$ remainder of composition $g(h)$ modulo f
- Let $g(x) = \sum_{i=0}^{m-1} g_i(x) \cdot x^{mi}$, where $\deg(g_i) < m$

$n = 4 = 2^2$

$g(x) = 2x^3 + 3x^2 + 4x + 1$

high degree: $x^2 \cdot (2x + 3)$, where $2x + 3 = g_1$

lower: $4x + 1 = g_0$

$\deg(g_0) = 1$

$\deg(g_1) = 1$

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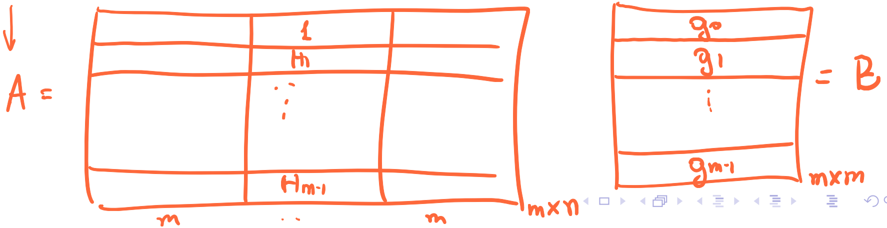
- Let $A \in \mathbb{F}^{m \times n}$ be the matrix whose rows are the coefficients of $1, H_1, H_2, \ldots, H_{m-1}$, and $B \in \mathbb{F}^{m \times m}$ be the matrix whose rows are the coefficients of $g_0, \ldots, g_{m-1}$.
- Compute $C = BA$ by doing m matrix multiplications of size $m \times m$



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- Let $A \in \mathbb{F}^{m \times n}$ be the matrix whose rows are the coefficients of $1, H_1, H_2, \ldots, H_{m-1}$, and $B \in \mathbb{F}^{m \times m}$ be the matrix whose rows are the coefficients of $g_0, \ldots, g_{m-1}$.
  Compute $C = BA$ by doing m matrix multiplications of size $m \times m$
- For $0 \leq i < m$, let $c_i(x) \in \mathbb{F}[x]$ be the polynomial corresponding to i^{th} row of C. Compute (using Horner's rule)

$$r(x) = \sum_{i=0}^{m-1} c_i \cdot (H_m)^i \mod f$$

and return r

Fast Algorithm in Picture

trying to beat $n \cdot M(n)$ → $m^2 \cdot M(m^2) \geq m^4$

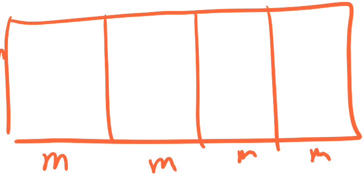
$H_i(x) = h^i \bmod f$ $0 \le i \le m$

(m+1 products of polynomials of degree $\le n$)

$(1+m) \cdot M(n)$

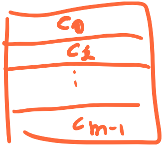
some savings here

$A :=$



BA $m \cdot MM(m)$

$C = BA$



$$r(x) = \sum_{i=0}^{m-1} c_i (H_m)^i \bmod f$$

Horner's rule $m \cdot M(n)$ to compute $r(x) \bmod f$

Total running time: $2(m+1)M(n) + m \cdot MM(m) = O(m M(m^2) + m^{3.8})$

$$g(x) = \sum_{i=0}^{m-1} g_i(x)\,(x^m)^i$$

$$g_i(x) = \sum_{j=0}^{m-1} g_{ij}\cdot x^j$$

$$H_i(x) = h(x)^i \bmod f \qquad 0 \le i \le m \quad (*)$$

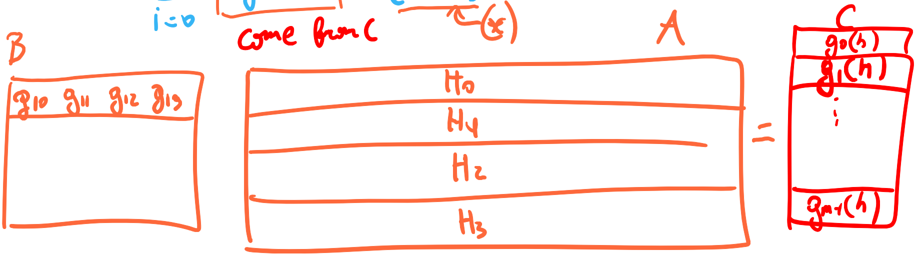
h^j

$g_{ij}\, H_j \bmod f$

$$g(h) = \sum_{i=0}^{m-1} g_i(h)\cdot (h^m)^i \equiv \bmod f$$

$$= \sum_{i=0}^{m-1} g_i(h)\cdot (H_m)^i$$

come from C ← (*)



Example of Fast Algorithm

- $f = x^4 - 1, g = x^3 + 1, h = x^2 + 1 \in \mathbb{F}_3[x]$

$m = 2$

$n = 4$

$$g(x) = 1 \cdot \underbrace{1}_{g_0(x)} + x^2 \cdot \underbrace{(x)}_{g_1(x)}$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}}_{C}$$

1	x	x^2	x^3
1	0	0	0
1	0	1	0

$h^0 \bmod f$

$h^1 \bmod f$

$h^2 \bmod f$

A comes from here

$g_0(h) = 1$ ←

$g_1(h) = 1 + x^2$ ←

1	x	x^2	x^3
1	0	0	0
1	0	1	0

$$r(x) = g_0(h)\cdot(H_2)^0 + g_1(h)\cdot(H_2)^1 \mod f$$

$$H_1 = h = x^2+1$$

$$H_2 = h^2 = x^4+2x^2+1 \equiv 2(x^2+1)$$

$$r(x) = 2(x^2+1)\cdot(1+x^2) + 1 \mod f$$

$$= 2\cdot 2(x^2+1) + 1$$

$$= 4x^2+5 \mod f$$ ← return.

Horner's rule

$$P(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$$

$$((((a_d x + a_{d-1}) x + a_{d-2}) x + a_{d-3}) x + \cdots$$

Conclusion

- Today we learned about Strassen's fast matrix multiplication algorithm
- Application to fast modular composition
- Next lectures: exploration of matrix multiplication exponent and fast linear algebra in "black box" model

References I

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Chapter 12