Lecture 11: Finding Short Vectors in a Lattice

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Overview

- Short Vectors in a Lattice
- Algorithm Idea: Find Good Basis
- Gram-Schmidt Orthogonalization
- Lenstra-Lenstra-Lovasz (LLL) Basis Reduction Algorithm
- Conclusion
- Acknowledgements

Short Vectors in a Lattice

- **Input:** linearly independent vectors $b_1, \ldots, b_n \in \mathbb{Z}^n$, bound $M \in \mathbb{N}$

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  $\lambda(\mathcal{L})$ is the length of the shortest vector in $\mathcal{L}$
- Today we will see a polynomial time algorithm when $M = 2^{\frac{n-1}{2}}$

(still low-bit complexity)

quite ok for our purposes

Observations on our Lattice Problem

- In previous lecture, we wrote the problem with input vectors

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where m, n could be distinct. Why **isn't** the problem from the previous slide *less general*?

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- If $m > n$, we can simply take a linearly independent subset of the vectors b_i which span the lattice.
- Given previous bullets, we can indeed assume that $m = n$.

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$$\begin{pmatrix} b_{11} & b_{21} & -- & b_{m1} \\ b_{12} & b_{22} & & \\ \vdots & & & \\ b_{1n} & & --- & b_{mn} \end{pmatrix}$$

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$$\mathcal{L} \ni c_k = b_k - \frac{b_{k1}}{g} \cdot c_1 \qquad \boxed{c_{k1} = 0}$$

Note that new basis also spans the *same lattice* $\mathcal{L}$ and $c_{k1} = 0$ for all $k > 1$.

$$c_1 = \sum_{k=1}^{m} a_k b_k$$

$$\begin{pmatrix} a_1 b_{11} + a_2 b_{21} + \cdots + a_m b_{m1} \\ * \\ * \\ \vdots \\ * \end{pmatrix} \quad a_1 b_{11} + a_2 b_{21} + \cdots + a_m b_{m1} = g$$

$$C = \left(\begin{array}{c|c} g & 0 \; 0 \; 0 \cdots 0 \\ \hline * & * \end{array}\right)$$

$$\mathcal{L}(b_1, \ldots, b_n) = \mathcal{L}(c_1, \ldots, c_n)$$

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    3. Repeat step (1) for $(c_2, \ldots, c_m)$ recursion

$$\left(\begin{array}{c|cccc} g & 0 & 0 & \cdots & 0 \\ \hline * & c_2 & c_3 & \cdots & c_m \end{array}\right)$$

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- Note that by the end of this process, we will have a matrix
$$M = \begin{pmatrix} A & 0 \end{pmatrix}$$
where $A \in \mathbb{Z}^{n \times n}$ is integral, full rank, and the column vectors of A span the same lattice $\mathcal{L}$.

Example

$$\begin{array}{c} \begin{pmatrix} 2 & 5 & 4 \\ 3 & -1 & 4 \end{pmatrix} \\ \begin{matrix} \uparrow & \uparrow & \uparrow \\ b_1 & b_2 & b_3 \end{matrix} \end{array}$$

$$1 = \gcd(2,5,4)$$

$$2 \cdot 0 + 5 \cdot 1 + 4 \cdot (-1)$$

$$c_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot 0 + \begin{pmatrix} 5 \\ -1 \end{pmatrix} \cdot 1 + (-1)\begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

$$c_2 = b_2 - \frac{5}{1} \cdot c_1 = \begin{pmatrix} 5 \\ -1 \end{pmatrix} - 5\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \end{pmatrix}$$

$$c_3 = b_3 - \frac{4}{1} \cdot c_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} - 4\begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -5 & 24 & 24 \end{pmatrix} \longmapsto \left(\underset{A}{\boxed{\begin{matrix} 1 & 0 \\ -5 & 24 \end{matrix}}} \begin{matrix} 0 \\ 0 \end{matrix} \right)$$

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Determinant of a Lattice

- Now that we clarified the assumption that $m = n$ and that $b_1, \ldots, b_n$ form a basis of $\mathbb{R}^n$, we can define an *invariant* of our lattice: the *determinant*

$$\det(\mathcal{L}) = \left| \det \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} \right|$$

$$B = \begin{pmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_n \\ | & | & & | \end{pmatrix} \qquad \det(\mathcal{L}) = |\det(B)|$$

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- The definition above is *basis independent*: if $(c_1, c_2, \ldots, c_n)$ is another basis for $\mathcal{L}$, we have that

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- Proof: invertible linear transformation taking one basis to another.

$$c_k = \sum_{j=1}^{n} A_{jk} b_j \qquad A_{kj} \in \mathbb{Z}$$

$$\begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & & A_{n1} \\ A_{21} & A_{22} & \ddots & \\ \vdots & & & \\ A_{n1} & & & A_{nn} \end{pmatrix}$$

$$B = C\tilde{A}$$

$$\underbrace{\begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} A_{11} & A_{12} & & A_{n1} \\ A_{21} & A_{22} & \ddots & \\ \vdots & & & \\ A_{n1} & & & A_{nn} \end{pmatrix}}_{A}$$

$$c_1 = A_{11} b_1 + A_{21} b_2 + \cdots + A_{n1} b_n$$

$$A_{ij} \in \mathbb{Z}$$

$$\underline{\det(C)} = \underline{\det(B)} \cdot \boxed{\det(A)}$$

$$= \det(BA) = \qquad \in \mathbb{Z}$$

$$\Rightarrow \det(B) \mid \det(C)$$

and $\det(C) \mid \det(B)$

$\Rightarrow \det(B) = \pm \det(C)$

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- Proof: invertible linear transformation taking one basis to another.
- To go from one basis to another, we can do elementary column operations, that is, if we have basis $b_1, \ldots, b_n$ then we can do

$$c_k = b_k - \alpha b_i, \ \alpha \in \mathbb{Z} \quad \text{and} \quad c_\ell = b_\ell \ \text{ for } \ell \neq k$$

Algorithm idea: a good basis will contain a short vector!

- Let's work this out for $n = 2$. Suppose we have $a, b \in \mathbb{Z}^2$ which form a basis for the lattice $\mathcal{L} = \mathbb{Z}a + \mathbb{Z}b$. Also, assume $\|a\| \leq \|b\|$.

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- Proof: let $z = \beta a + \gamma b$, where $\beta, \gamma \in \mathbb{Z}$. Can assume $\beta, \gamma \neq 0$

$\gamma = 0 \implies z = \beta a \implies \|z\| = |\beta| \cdot \|a\|$
$\geq \|a\|$

$\beta = 0 \implies z = \gamma b \implies \|z\| \geq \|b\| \geq \|a\|$

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- Case 1: $\beta > \gamma$ $\quad \beta > 0$

$$(a+b)^2 = \|a\|^2 + \cancel{\|b\|^2} + 2\langle a, b\rangle \geq \cancel{\|b\|^2} = \langle b, b\rangle$$

$$\Rightarrow 2\langle a, b\rangle \geq -\langle a, a\rangle$$

$$\|z\|^2 = \langle \beta a + \gamma b, \beta a + \gamma b\rangle = \beta^2\|a\|^2 + \gamma^2\|b\|^2 + 2\beta\gamma\langle a, b\rangle$$

$$\geq \beta^2\|a\|^2 + \gamma^2\|b\|^2 - \beta\gamma\|a\|^2 = \beta(\beta-\gamma)\|a\|^2 + \gamma^2\|b\|^2$$

$$\geq \underbrace{\beta}_{>0}\underbrace{(\beta-\gamma)}_{>0}\|a\|^2 \geq \|a\|^2$$

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$\|a+b\|^2 \geq \|a\|^2$ similar to previous slide

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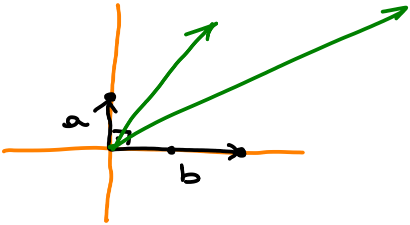

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$$\begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$$

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- "Close enough" to orthogonal does it!

Looking at Counterexample

$$B = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix} \qquad U = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \text{ orthogonal}$$

$$|\det(B)| = |\det(U)| \quad \text{invariant}$$

$$|0 \cdot (-1) - 1 \cdot 3| = 3 = \underbrace{|\det(U)|}$$

$$= \|u_1\| \cdot \|u_2\|$$

$$U \cdot U^T = \det(U)^2 \cdot I \quad (U \text{ orthogonal})$$

$$\begin{pmatrix} \|u_1\|^2 & 0 \\ 0 & \|u_2\|^2 \end{pmatrix}$$

$\|u_1\| \cdot \|u_2\| = 3$ $\qquad U \qquad$ $B = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}$

$u_1, u_2 \in \mathcal{L}\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$

$\Rightarrow u_1, u_2 \in \mathbb{Z}^2$

$\|u_1\|^2 \cdot \|u_2\|^2 = 9 \Rightarrow$ $\begin{array}{l} \|u_1\| = 3 \\ \|u_2\| = 1 \\ \hline e_1, e_2 \end{array}$

$\|u_1\| = \|u_2\| = \sqrt{3}$

Gauss' Reduction Algorithm

- The LLL algorithm is generalization of 2D basis reduction due to Gauss
- Idea: given two vectors u, v, s.t. $\|u\| \leq \|v\|$ subtract off as much of u's projection from v, while staying in the lattice

Making u, v "as orthogonal as possible"

$\|v - \alpha u\|$ smallest $\alpha \in \mathbb{Z}$

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- Proof: if $\beta = \frac{\langle u, v \rangle}{\|u\|}$, take $\alpha \in \mathbb{Z}$ closest to β. Thus $|\alpha - \beta| \leq 1/2$

$$|\langle \underbrace{v - \alpha u}_{v - \beta u + (\beta - \alpha) u}, u \rangle| = |\langle \cancel{v - \beta u, u} \rangle + \langle (\beta - \alpha) u, u \rangle| \leq \frac{1}{2}\|u\|^2$$

?

$v - \beta u \perp u$

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$$|\langle v - \alpha u, u \rangle| = |\langle v - \beta u, u \rangle + \langle (\beta - \alpha) u, u \rangle| \leq \frac{1}{2}\|u\|^2$$

- If $\|v - \alpha u\| \geq \|u\|$ stop. Otherwise swap the vectors and continue.

Gauss' Reduction Algorithm

- The LLL algorithm is generalization of 2D basis reduction due to Gauss
- Idea: given two vectors u, v, s.t. $\|u\| \leq \|v\|$ subtract off as much of u's projection from v, while staying in the lattice
- There is $\alpha \in \mathbb{Z}$ such that

$$|\langle v - \alpha u, u \rangle| \leq \frac{1}{2} \|u\|^2$$

- Proof: if $\beta = \dfrac{\langle u, v \rangle}{\|u\|}$, take $\alpha \in \mathbb{Z}$ closest to β. Thus $|\alpha - \beta| \leq 1/2$

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- If $\|v - \alpha u\| \geq \|u\|$ stop. Otherwise swap the vectors and continue.
- Note that at each iteration we are decreasing the norm of the smallest basis vector. When we cannot decrease further, previous slide gives us that u is the shortest vector!

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Orthogonal Bases and Short Vectors

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$$\underline{u_k} = b_k - \sum_{i=1}^{k-1} \underbrace{\frac{\langle b_k, u_i \rangle}{\|u_i\|^2} \cdot u_i}_{\text{projection of } b_k \text{ onto } u_i}$$

$\langle u_k, u_i \rangle = 0$
$i < k$

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rational #s (non integers)

$$u_k = b_k - \sum_{i=1}^{k-1} \boxed{\frac{\langle b_k, u_i \rangle}{\|u_i\|^2}} \cdot u_i$$

- Orthogonal basis *not necessarily* a basis for our lattice!

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- If don't change the order but make some $b_k = b_k + \alpha b_j$ with $j < k$ the GSO basis stays the same
- If input basis is *integral* (or rational) then the output basis is *rational*

Shortest vector & Gram-Schmidt Orthogonalization (GSO)

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- By GSO, we can write $b_k = \sum_{i=1}^{k} \mu_{ki} \cdot u_i$, with $\mu_{kk} = 1$
- Thus, if $\alpha_t \neq 0$ and $\alpha_\ell = 0$ for all $\ell > t$:

$$v = \beta_1 u_1 + \cdots + \underline{\beta_t} u_t$$

With $\boxed{\beta_t = \cancel{1}\ \alpha_t}$, as no other u_i depends on u_t.

$b_1, \ldots, b_{t-1}$ don't depend on u_t

$$v = \alpha_1 b_1 + \cdots + \alpha_t b_t = \sum_{i=1}^{t} \beta_i u_i$$

$b_t = u_t + (-)$

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- And the norm is given by:

$$\|v\| = \underbrace{|\beta_1| \cdot \|u_1\| + \cdots +}_{\geq 0} \underbrace{|\beta_t|}_{\geq 1} \cdot \|u_t\| \geq \underbrace{\|u_t\|}_{\in \mathrm{GSO}}$$

Reduced Basis

- Now we are ready to define what a "good basis" is:
- Let $(u_1, \ldots, u_n)$ be the GSO basis from $(b_1, \ldots, b_n)$

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- A basis $(b_1, \ldots, b_n)$ is a *reduced basis* if
  1. each $|\mu_{ki}| \leq 1/2$ when $i \neq k$ ← orthogonality of b_i's
  2. For each k,

$$\|u_k\|^2 \leq \frac{4}{3} \cdot \|u_{k+1} + \mu_{(k+1)k} u_k\|^2$$

GSO basis does not have "spikes"

$$b_k = u_k + \sum_{i<k} \underline{\mu_{ki}}\, u_i$$

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  $$\|u_k\|^2 \leq \frac{4}{3} \cdot \|u_{k+1} + \mu_{(k+1)k} u_k\|^2$$
- The LLL basis reduction algorithm will simply construct a reduced basis iteratively, much like Gauss' reduction algorithm.

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GSO $(u_1, \ldots, u_n)$

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$|\mu_{ki}| > \frac{1}{2}$ (b_i, b_k) Gauss' reduction

$(i<k)$

$$\left.\begin{array}{l} b_k \leftarrow b_k - \alpha b_i \\ |\mu_{ki}| \leq \frac{1}{2} \end{array}\right\}$$

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- We will now take a deeper look into the first routine

Step 1 – Gauss Reduction

- Given basis $(b_1, \ldots, b_n)$ $\mathcal{L}$ with GSO basis $(u_1, \ldots, u_n)$, we can get a new basis $(c_1, \ldots, c_n)$ $\mathcal{L}$ where

$$c_k = \sum_{i=1}^{k} \gamma_{ki} u_i \quad \text{with} \quad |\gamma_{ki}| \leq 1/2 \qquad i < k$$

$(c_1, \ldots, c_n)$ has <u>some</u> GSO basis

as $(b_1, \ldots, b_n)$

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- If $(b_1, \ldots, b_n)$ does not have desired property, take maximum pair $\underbrace{(k, i)}$ such that $|\mu_{ki}| > 1/2$.

$$\underline{b'_k} := b_k - \alpha b_i \quad \text{from Gauss reduction}$$

$$\Longrightarrow \quad |\mu_{ki}| \leq 1/2$$

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only changes $\mu_{k''i''}$ smaller pairs

- Why maximum? Because we don't mess up the higher μ's (but we may mess up the lower ones)

$\mu_{k'i'}$ $(k', i') > (k, i)$

not affected

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- After we go through all pairs (k, i) in decreasing order, the new coefficients γ_{ki} will satisfy 1

do this $O(n^2)$ times

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- We need to prove that our algorithm will terminate, and will do so quickly

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  - Step 1 does not change the GSO basis, so D is unchanged

$$D(b_1, \ldots, b_n) = D(c_1, \ldots, c_n)$$

Gauss reductions didn't change the GSO basis

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- Upper bound on $D(b_1, \ldots, b_n)$:

$$D(b_1, \ldots, b_n) \leq (\max_i \|u_i\|)^{n^2} < \underbrace{p(\|b_i\|)^{n^2}}_{\exp(n^2, b)}$$

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$$D(b_1, \ldots, b_n) \leq (\max_i \|u_i\|)^{n^2} \leq \exp(n)$$

- Lower bound: let $B = (b_1 b_2 \cdots b_n)$

$$1 \leq \underbrace{\det(B^T B)}_{\text{integer} > 0} = \prod_{i=1}^{n} \|u_i\|^2$$

$$b_k = u_k + \sum_{i<k} \mu_{ki} u_i$$

$$U = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

$$B = \begin{pmatrix} u_1 & \cdots & u_n \end{pmatrix} \underbrace{\begin{pmatrix} 1 & \mu_{21} & \mu_{31} & & \\ 0 & 1 & \mu_{32} & & \mu_{ki} \\ 0 & 0 & 1 & \ddots & \\ \vdots & \vdots & 0 & & \ddots \\ 0 & 0 & \vdots & & 1 \end{pmatrix}}_{A}$$

$$B = U \cdot A \to \text{upper triangular, 1's diagonal}$$

$$B^T B = A^T U^T U A \Rightarrow \det(B^T B) = \det(A^T A) \cdot \det(U^T U)$$

$$\det(B^T B) = \underbrace{\det(A^T A)}_{\det(A)^2 \,=\, 1} \cdot \underbrace{\det(U^T U)}_{\prod_{i=1}^{n} \|u_i\|^2}$$

$$\det(B^T B) = \underbrace{\det(B)^2}_{> 0}$$

integer

$\Rightarrow\ \geq 1$

Finding Short Vector

- If $(b_1, \ldots, b_n)$ is a reduced basis of $\mathcal{L}$, then

$$\|b_1\| \leq 2^{\frac{n-1}{2}} \lambda(\mathcal{L})$$

where $\lambda(\mathcal{L})$ is the length of the shortest vector in $\mathcal{L}$

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- By reduced property of our basis, if $(u_1, \ldots, u_n)$ is the GSO basis we have:

$$\begin{aligned}
\|u_k\|^2 &\leq \frac{4}{3} \cdot \|u_{k+1} + \mu_{(k+1)k} u_k\|^2 \\
&= \frac{4}{3} \cdot \|u_{k+1}\|^2 + \frac{4}{3} \cdot \boxed{\mu^2_{(k+1)k}} \cdot \|u_k\|^2 \\
&\leq \frac{4}{3} \cdot \|u_{k+1}\|^2 + \frac{1}{3} \cdot \|u_k\|^2 \\
&\Rightarrow \|u_k\|^2 \leq 2\|u_{k+1}\|^2
\end{aligned}$$

$|\mu_{(k+1)k}| \leq \frac{1}{2}$

$\frac{2}{3} \|u_k\|^2 \leq \frac{4}{3} \|u_{k+1}\|^2$

Finding Short Vector

- If $(b_1, \ldots, b_n)$ is a reduced basis of $\mathcal{L}$, then

$$\|b_1\| \leq 2^{\frac{n-1}{2}} \lambda(\mathcal{L})$$

where $\lambda(\mathcal{L})$ is the length of the shortest vector in $\mathcal{L}$

- By reduced property of our basis, if $(u_1, \ldots, u_n)$ is the GSO basis we have:

$\|b_1\|^2$

$\|$

$\|u_1\|^2 \leq 2^{k-1} \|u_k\|^2$

$$\begin{aligned}
\|u_k\|^2 &\leq \frac{4}{3} \cdot \|u_{k+1} + \mu_{(k+1)k} u_k\|^2 \\
&= \frac{4}{3} \cdot \|u_{k+1}\|^2 + \frac{4}{3} \cdot \mu_{(k+1)k}^2 \cdot \|u_k\|^2 \\
&\leq \frac{4}{3} \cdot \|u_{k+1}\|^2 + \frac{1}{3} \cdot \|u_k\|^2 \\
&\Rightarrow \|u_k\|^2 \leq 2\|u_{k+1}\|^2
\end{aligned}$$

← induction

- Then our lemma on GSO basis and shortest vector gives us $\underline{\|b_1\|}^2 \leq \underline{\min_k\{2^{k-1}\|u_k\|^2\}} \leq 2^{n-1} \cdot \underbrace{\min_k \|u_k\|^2}_{\text{GSO}} \leq 2^{n-1} \cdot \underline{\lambda(\mathcal{L})}^2$

Proof Details

- Short Vectors in a Lattice
- Algorithm Idea: Find Good Basis
- Gram-Schmidt Orthogonalization
- Lenstra-Lenstra-Lovasz (LLL) Basis Reduction Algorithm
- Conclusion
- Acknowledgements

Conclusion

In today's lecture, we learned

- Finding short vector in a lattice
- Finished proof of factoring algorithm over $\mathbb{Z}[x]$
- LLL algorithm is useful way beyond factoring!
    1. breaking cryptosystems
    2. finding simultaneous Diophantine approximations
    3. refutation of Mertens' conjecture
- Great final projects to explore here!

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- Lectures 10 and 11 from Madhu's notes
  `http://people.csail.mit.edu/madhu/FT98/`