CS 487 / . . .
Introduction to Symbolic Computation

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Linear recurrences with polynomial coefficients

($n!$ and generalizations)

D-finite series

Def.

- A power series $f(x)$ is **D-finite** if there exists a linear differential equation with polynomial coefficients such that

$$q_d(x)f^{(d)} + q_{d-1}(x)f^{(d-1)} + \cdots + q_0(x)f = 0.$$

- Equivalently, we can take rational functions as coefficients.

Examples.

- polynomials,
- rational functions,
- algebraic series (e.g., $\sqrt{1+x^2}$)
- exp, sin, cos,
- a lot more

P-recursive sequences

Def.

- A sequence u_n is **P-recursive** if it satisfies a recurrence with polynomial coefficients

$$p_d(n)u_{n+d} + p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n = 0$$

Examples.

- constant sequences,
- recurrences with constant coefficients,
- factorial, and generalizations.

Remark: matrix recurrence

$$p_d(n)u_{n+d} + p_{d-1}(n)u_{n+d-1} + \cdots + p_0(n)u_n = 0$$

means that

$$u_{n+d} = -\frac{p_{d-1}(n)}{p_d(n)}u_{n+d-1} - \cdots - \frac{p_0(n)}{p_d(n)}u_n$$

so

$$\begin{bmatrix} u_{n+d} \\ u_{n+d-1} \\ \vdots \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} -\frac{p_{d-1}(n)}{p_d(n)} & -\frac{p_{d-2}(n)}{p_d(n)} & \cdots & -\frac{p_0(n)}{p_d(n)} \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} u_{n+d-1} \\ u_{n+d-2} \\ \vdots \\ u_n \end{bmatrix}$$

Equivalence

Theorem.

- The power series

$$f = \sum_{i \geqslant 0} f_i x^i$$

  is D-finite if and only if the sequence (f_i) is P-recursive.

Examples.

- recurrence with constant coefficients $\iff$ rational power series.
- $f_i = 1/i!$ $\iff$ exponential

Proof for the exponential

Suppose that f is a solution of

$$f' = f.$$

(We know that f is the exponential.) With

$$f = \sum_{i \geqslant 0} f_i x^i,$$

we get

$$f' = \sum_{i \geqslant 0} (i+1) f_{i+1} x^i.$$

So

$$(i+1) f_{i+1} = f_i.$$

Sketch of proof in general

In general, with

$$f = \sum_{i \geqslant 0} f_i x^i,$$

we get

$$f' = \sum_{i \geqslant 0} (i+1) f_{i+1} x^i \quad \text{and} \quad f'' = \sum_{i \geqslant 0} (i+1)(i+2) f_{i+2} x^i, \quad \ldots$$

Multiplying by a monomial **shifts** the coefficients:

$$x^\ell f' = \sum_{i \geqslant 0} (i+1) f_{i+1} x^{i+\ell} = \sum_{i \geqslant \ell} (i-\ell+1) f_{i-\ell+1} x^i, \quad \ldots$$

So extracting coefficients gives a recurrence on the f_i.

Converse on an example

Consider the **factorial**

$$f_i = i!, \quad \text{so that} \quad f_{i+1} = (i+1)f_i.$$

Let $f = \sum_{i \geqslant 0} f_i x^i$.

Multiply by x^{i+1} and sum over all $i \geqslant 0$.

$$\sum_{i \geqslant 0} f_{i+1} x^{i+1} = f - 1 \quad \text{and} \quad \sum_{i \geqslant 0} (i+1) f_i x^{i+1} = x(xf' + f).$$

So

$$\boldsymbol{x^2 f' + (x-1)f = -1} \quad \text{or} \quad \boldsymbol{x^2 f'' + (3x-1)f' - f = 0}.$$

Our questions

1. Computing one term in a P-recursive sequence

- binary splitting
- baby steps / giant steps

2. Computing several terms

- unroll the recurrence
- solve the differential equation using Newton iteration

Examples

1. Compute the first 100 coefficients c_i of $bm(x+1)^{10000}$
We know they are binomial coefficients, so we get

$$c_{i+1} = \frac{D-i}{i+1} c_i$$

2. Compute the first 100 coefficients d_i of $(x+2)^{10000}(x+1)^{1000}$
The generating series $S = \sum_{i \geqslant 0} d_i x^i$ satisfies

$$\frac{S'}{S} = 10000 \frac{1}{x+2} + 1000 \frac{1}{x+1}$$

which gives

$$(i-1)d_{i-1} + 3id_i + 2(i+1)d_{i+1} = 11000d_{i-1} + 12000d_i$$

Computing one term

Binary splitting

This is the method you want to use when the **coefficients size** matters

- quasi-optimal algorithms exist, taking bit-size into account
- not useful modulo p

Example: factorial.

- We write $\mathbf{M}_{\mathbb{Z}}(n)$ for the cost of multiplying integers of size n.
- The factorial $n!$ has about $n \log n$ digits.

Prop.

- Using **binary splitting**, one can compute $n!$ in $O(\mathbf{M}_{\mathbb{Z}}(n \log n) \log n)$ bit operations.

The algorithm in a nutshell

It boils down to computing $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots$ in a clever way.

Naive:

- $2 = 1 \cdot 2$
- $6 = 2 \cdot 3$
- $24 = 6 \cdot 4$
- $120 = 24 \cdot 5$
- $720 = 120 \cdot 6$
- $5040 = 720 \cdot 7$
- $40320 = 5040 \cdot 8$
- $362880 = 40320 \cdot 9$

Consequence: quadratic time!

The algorithm in a nutshell

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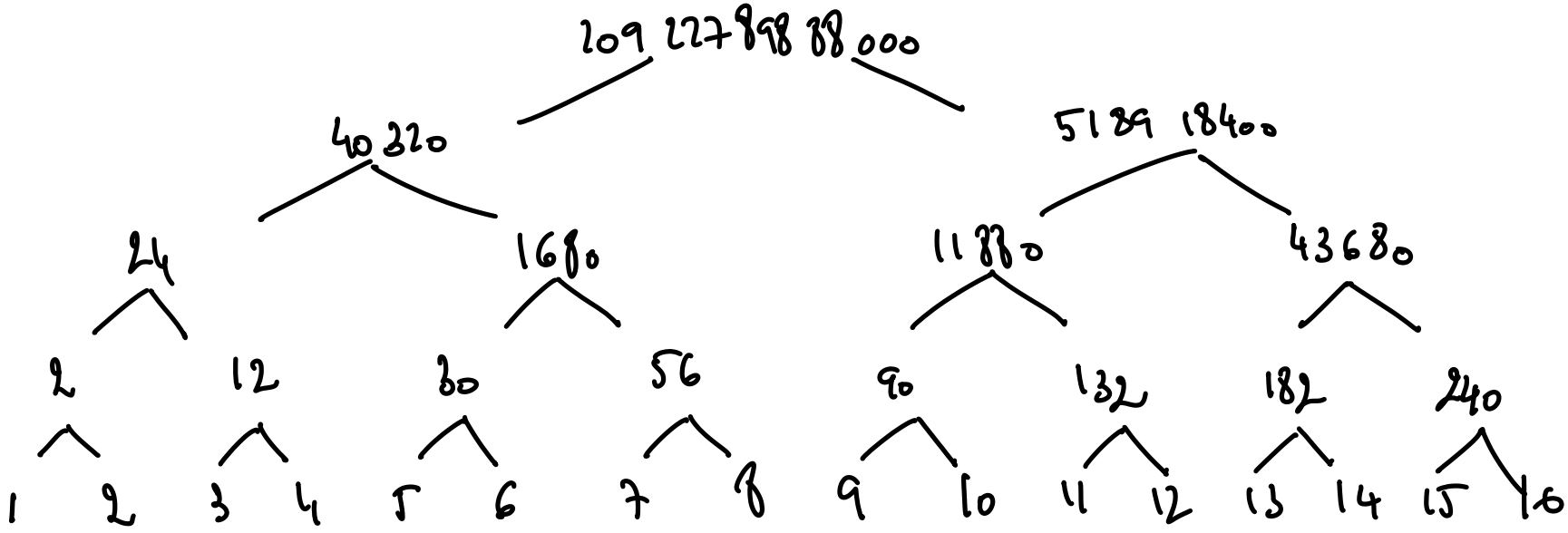

Splitting

Let $P(a,b) = a(a+1)\cdots b$, so that we want $P(1,n)$.

Binary splitting:

$$P(a,b) = P(a,m)P(m,b) \quad \text{with} \quad m = \lfloor (a+b)/2 \rfloor.$$

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Cost:

$$\begin{aligned} C(a,b) &= C(a,m) + C(m,b) + \mathsf{M}_{\mathbb{Z}}(\log P(m,b)) \\ &\leqslant 2C(m,b) + \mathsf{M}_{\mathbb{Z}}(\log P(m,b)). \end{aligned}$$

Splitting

Let $P(a,b) = a(a+1)\cdots b$, so that we want $P(1,n)$.

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The splitting scheme

- $C(1,n) \leqslant 2C(n/2,n) + \mathsf{M}_{\mathbb{Z}}(\log P(n/2,n))$

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- $C(1,n) \leqslant 2C(n/2,n) + \mathsf{M}_{\mathbb{Z}}(\log P(n/2,n))$
- $C(n/2,n) \leqslant 2C(3n/4,n) + \mathsf{M}_{\mathbb{Z}}(\log P(3n/4,n))$

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- $C(1,n) \leqslant 2C(n/2,n) + \mathsf{M}_{\mathbb{Z}}(\log P(n/2,n))$
- $C(n/2,n) \leqslant 2C(3n/4,n) + \mathsf{M}_{\mathbb{Z}}(\log P(3n/4,n))$
- $C(3n/4,n) \leqslant 2C(7n/8,n) + \mathsf{M}_{\mathbb{Z}}(\log P(7n/8,n))$

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- $C(1,n) \leqslant 2C(n/2,n) + \mathsf{M}_{\mathbb{Z}}(\log P(n/2,n))$
- $2C(n/2,n) \leqslant 4C(3n/4,n) + 2\mathsf{M}_{\mathbb{Z}}(\log P(3n/4,n))$

Splitting

Let $P(a,b) = a(a+1)\cdots b$, so that we want $P(1,n)$.

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- $C(1,n) \leqslant 2C(n/2,n) + \mathsf{M}_{\mathbb{Z}}(\log P(n/2,n))$
- $2C(n/2,n) \leqslant 4C(3n/4,n) + 2\mathsf{M}_{\mathbb{Z}}(\log P(3n/4,n))$
- $4C(3n/4,n) \leqslant 8C(7n/8,n) + 4\mathsf{M}_{\mathbb{Z}}(\log P(7n/8,n))$

Solving the recurrence

These equalities give (for any $k \leqslant \log(n)$)

$$C(1,n) \;\leqslant\; \mathbf{2^k}\, C(n - \frac{n}{\mathbf{2^k}}, n) \;+ \sum_{j=1}^{k} \mathbf{2^{j-1}}\, \mathsf{M}_{\mathbb{Z}}(\log P(n - \frac{n}{\mathbf{2^j}}, n)).$$

Simplifications

- remember that

$$P(n - \frac{n}{2^j}, n) = (n - \frac{n}{2^j}) \cdots n \leqslant n^{n/2^j}$$

- so its log is $\leqslant \frac{n}{2^j} \log n$,
- so its contribution is $\leqslant \mathsf{M}_{\mathbb{Z}}(n \log n)$
  (using $\mathsf{M}_{\mathbb{Z}}(t/2) \leqslant \frac{1}{2}\mathsf{M}_{\mathbb{Z}}(t)$)

Solving the recurrence

Putting everything together gives

$$C(1,n) \;\leqslant\; 2^k\, C(n - \frac{n}{2^k}, n) \;+\; k\mathsf{M}_{\mathbb{Z}}(n \log n).$$

We stop the recursion for $k = \log n$, which gives

$$C(1,n) \;\in\; O(\mathsf{M}_{\mathbb{Z}}(n \log n) \log n).$$

Second example: computing $e = \exp(1)$

The sequence

$$e_n = \sum_{k=0}^{n} \frac{1}{k!}$$

converges to e, and $0 \leqslant e - e_n \leqslant \frac{1}{n!}$.

Consequence

- To compute m digits of e, compute e_n, with

$$n \approx \frac{m}{\log m}$$

The recursion

The sequence $f_n = 1/n!$ satisfies the recurrence

$$(n+1)f_{n+1} = f_n.$$

Because $e_{n+1} - e_n = f_{n+1}$, we get

$$(n+1)(e_{n+1} - e_n) = (n+1)f_{n+1} = f_n = e_n - e_{n-1},$$

which becomes

$$\begin{bmatrix} e_{n+1} \\ e_n \end{bmatrix} = \frac{1}{n+1} \begin{bmatrix} n+2 & -1 \\ n+1 & 0 \end{bmatrix} \begin{bmatrix} e_n \\ e_{n-1} \end{bmatrix} = \frac{1}{n+1} M(n) \begin{bmatrix} e_n \\ e_{n-1} \end{bmatrix}.$$

So to compute e_{n-1} and e_n, we actually compute

$$\frac{1}{n!} M(n-1) \cdots M(1).$$

Same thing as the factorial!

Example

Take $n = 30$; then $M(n-1)\cdots M(1)$ is

$$\begin{bmatrix} 1412915423782455596182116504504732 & -5906315583646633144095602165504732 \\ 1412915423782455596182116504504731 & -5906315583646633144095602165504731 \end{bmatrix}$$

and $n! = 822283865417792281772556288000000$.

Our approximation to $\exp(1)$ is the first entry of

$$\frac{1}{n!}M(n-1)\cdots M(1)\begin{bmatrix}2\\1\end{bmatrix} = \frac{5587998223000619694886681981376183}{2055709663544480704431390720000000}.$$

Gives **117** correct bits.

Remark: works for cos, sin, ..., all D-finite functions.

Baby steps / giant steps

This is the method to use when coefficient size does not matter.

Prop.

- Consider u_n defined by a recurrence of order d with coefficients of degree p.
- Then the nth term can be computed in $O(\mathsf{M}(\sqrt{n})\log n)$, where the big-Oh depends on d and p.

Example

- The sequence $u_{n+1} = (n+1)u_n$, computed modulo an integer N.
- This leads to the best deterministic, proved algorithm for factoring integers.

Preliminaries

Evaluation and interpolation

- Given a polynomial P of degree $m-1$, and m evaluation points $a_0, \ldots, a_{m-1}$ one can compute

$$P(a_0), \ \ldots, \ P(a_{m-1})$$

  in $\boldsymbol{O(\mathbf{M}(m)\log(m))}$ operations.
- Conversely, given the values, one can recover P in the same cost.

Main ideas

- Divide-and-conquer: replace the original problem by the evaluation of a polynomial P_0 at the first half of the points and a polynomial P_1 at the second half.
- Cost: $C(n) \leqslant 2C(n/2) + O(\mathsf{M}(n))$.

The example of the factorial

Consider the sequence $u_{n+1} = (n+1)u_n$, $u_0 = 1$.

To compute u_n, let $\boldsymbol{m = \sqrt{n}}$ and introduce

$$P = (x+1)\cdots(x+m).$$

Then u_n is given by

$$u_n = P(0)\,P(m)\,P(2m)\,\cdots\,P((m-1)m).$$

Algorithm

- Compute P (divide-and-conquer) $\quad O(\mathbf{M}(m)\log m)$
- Evaluate it a $0, m, \ldots, (m-1)m$ $\quad O(\mathbf{M}(m)\log m)$
- Multiply the values $\quad O(m)$

Application to factoring integers

Suppose you want to factor $p \in \mathbb{N}$ into primes.

- It's enough to find all prime factors $< \sqrt{p}$.
- Testing one number mod p costs $\boldsymbol{O((\log p)^{O(1)})}$.
- So naive cost $\boldsymbol{O(\sqrt{p}(\log p)^{O(1)})}$

Better: let $n = \sqrt{p}$ and $m = \sqrt{n}$, and compute the slices

$$a_0 = 1 \cdots m \bmod p,\ a_1 = (m+1) \cdots (2m) \bmod p, \ldots a_{m-1} = (m^2-m+1) \cdots m^2 \bmod p,$$

- cost almost linear in $\sqrt[4]{p}$.
- if $\gcd(a_i, p) = 1$, no divisor in the slice i.
- as soon as you found $\gcd(a_i, p) \neq 1$, test all elements in a_i.
- repeat.