Lecture 23: Zero-Knowledge Proofs

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Overview

- Why Zero Knowledge?
- Zero-Knowledge Proofs
- Conclusion
- Acknowledgements

Cyptography

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  - Can Alice convince Bob that she gave right file without giving any more *knowledge* beyond that she gave right file?

Zero-Knowledge Proofs

Proofs in which the verifier gains *no knowledge* beyond the validity of the assertion.

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- In both cases Alice conveyed *information*!

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$\mathcal{C}$

Prover $\xrightarrow{\pi}$ Verifier

$V(\mathcal{C}, \pi)$ → valid / not valid

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- Verifier *does not trust* prover. Otherwise no need to verify proof!

Example: NP (Efficient Verifiable Proofs)

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- In this setting, verifier *learns the proof*!

because the prover sent the whole proof to the verifier!

$P \neq NP$ then verifier alone could not have given right answer on its own

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  - A claim $\mathcal{C}$:= $\boxed{N \text{ is a product of two primes}}$ is given

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  - Verifier computes $p \cdot q$ and checks that p, q are prime.
    Checking validity of proof (*deterministic, polynomial time algorithm*)

Can multiply in P

Can check that number is prime in P

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Example: Graph Isomorphism

$V = [n]$ $G_0(V, E_0)$

$G_1(V, E_1)$

- Setup:
  - A claim $\mathcal{C} :=$ graphs G_0, G_1 are isomorphic

G_0 and G_1 are isomorphic iff there is a permutation of the vertices $\rho \in S_n$

s.t. $\{i, j\} \in E_0 \iff \{\rho(i), \rho(j)\} \in E_1$

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- In this setting, verifier *learns the isomorphism* (i.e., the proof)!

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- In the end, we will see a (zero-knowledge) proof for graph isomorphism as follows:

  Alice: I will not give you an isomorphism, but I will prove that I could give you one, if I wanted to.

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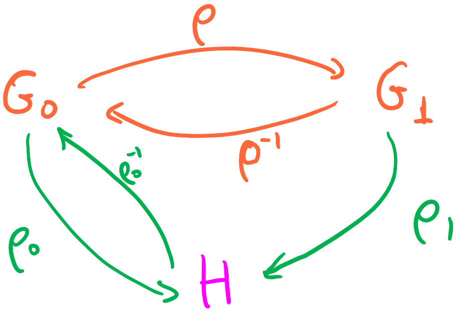

H is a random permutation of G_0

$H = \sigma(G_0)$

$$\boxed{\begin{aligned} \sigma &= \rho_0 \\ \rho_1 &= \rho_0 \circ \rho \end{aligned}}$$

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prover sends graph H to verifier

verifier sends bit b to prover

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$V(H, b, \rho_b)$

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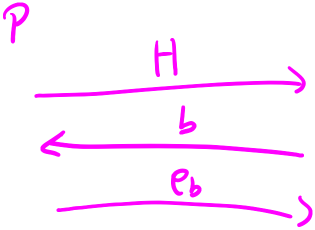

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  - Claim is *true* $\Rightarrow$ prover can always give isomorphism!
  - Claim is *false* $\Rightarrow$ can catch bad proof with probability $= 1/2$

P V

$H \sim G_0 \xrightarrow{H}$

$\xleftarrow{1}$

ρ_1 doesn't exist

because G_0, G_1 are not isomorphic

Verification always succeeds

$G_0 \xrightarrow{\sigma} H$, $H \xrightarrow{\sigma^{-1}} G_0$ $\quad G_1$

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  - Claim is *false* $\Rightarrow$ can catch bad proof with probability $= 1/2$
  - Can amplify probability of catching bad proof by repeating protocol above!
- How can we model the fact that verifier does not gain knowledge?!

Simulation!

Can do it by itself

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- Simulation $\Rightarrow$ V gained no new information!

$$\xrightarrow{\quad H \quad}$$
$$\xleftarrow{\quad b \quad}$$
$$\xrightarrow{\quad \rho \quad}$$

(H, b, ρ)
random variable

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A prover P is *perfect zero-knowledge* for language L if for every polynomial time, randomized verifier V^*, there is a randomized algorithm M^* such that for every $x \in L$ the following random variables are identically distributed:

- $\langle P, V^* \rangle(x)$, that is, output of interaction between prover P and verifier V^* on input x random variable of proof
- $M^*(x)$, that is, output of algorithm M^* (simulation) on input x

random variable of simulation

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- The above captures the idea that V^* is not gaining any extra computational power by interacting with P, since same output could have been generated by M^*

Perfect Zero Knowledge Proof[2]

- Previous definition is a bit too strict to be useful, so we relax it.[1]
- We will allow simulator to fail with small probability (denoted by outputting $\perp$)

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Perfect Zero Knowledge Proof[2]

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Definition (Perfect Zero Knowledge)

A prover P is *perfect zero-knowledge* for language L if for every polynomial time, randomized verifier V^*, there is a randomized algorithm M^* such that for every $x \in L$ the following holds:

1. With probability $\leq 1/2$, $M^*(x) = \perp$ simulator fails $\leq \frac{1}{2}$ of time
2. Conditioned on $M^*(x) \neq \perp$, the following variables are identially distributed:
   - $\langle P, V^* \rangle(x)$, that is, output of interaction between prover P and verifier V^* on input x
   - $M^*(x)$, that is, output of algorithm M^* (simulation) on input x

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- If $b \neq 0$ then output $\perp$
- Otherwise simulator gives isomorphism ρ^{-1}
- Simulator checks that $\rho^{-1}(H) = G_0$ always passes

if $b=0$

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- Simulation $\Rightarrow$ perfect zero knowledge for our prover P!
- Note that whenever we don't fail, we output same distribution as the original protocol!

$P, V \quad (H, 0, \rho_0)$ $\qquad$ $M \quad (H, 0, \rho_0)$

$H = \rho_0(G_0)$ $\qquad$ $H = \rho_0(G)$

random variable ρ_0 $\longleftrightarrow$ picking ρ.

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- We saw today how the power of interaction can be used to verify validity of "proofs" without conveying information about it
- Has applications in
  - Modern cryptography
  - Credit Cards
  - Passwords
  - Complexity Theory (can use zero-knowledge to construct complexity classes)
  - Used in cryptocurrencies (validate transactions without giving details about transactions)

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    `https://inst.eecs.berkeley.edu/~cs276/fa20/slides/lec14.pdf`