Lecture 22: Distributed Algorithms

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Overview

- Administrivia
- Distributed Computing: The Models
- Consensus with Byzantine Failures
- Conclusion
- Acknowledgements

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Evaluation will be open until August 5th.

- This would really help me figuring out what worked and what didn't for the course
- And let the school know if I was a good boy this term!
- Teaching this course is also a learning experience for me :)

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- Challenges in this setting:
  - Concurrent Activity
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  - Failure and recovery of processors or channels
- Many models
  - *Memory & Communication*: shared memory, message-passing
  - *Timing*: synchronous (rounds), asynchronous, partially synchronous (bounds on message delay, processor speeds, clock rates)
  - *Failures*: processor (stop, Byzantine), communication (message loss/altered), system state corruption

have to specify one for each of these categories

Synchronous Model

- processors are vertices of directed graph
  - *Memory*: each processor has its own memory
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↳ $\Sigma = \{0,1\} \cup \{\perp\}$

$\Sigma = \{1, \ldots, n\}$

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- For each vertex $i \in [n]$, a processor consists of:
  - S_i = non-empty set of states
  - σ_i = a start state
  - $\mu_i : S_i \times out_i \to \Sigma \cup \{\perp\}$ — Message function
  - $\tau_i : \underbrace{S_i}_{\text{current state}} \times \underbrace{(\Sigma \cup \{\perp\})}_{\text{all incoming messages of the round}}{}^{in_i} \to S_i$ — Transition function

$out_i \leftarrow$ set of outgoing edges of i

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- Complexity Measure: *number of rounds* (*total data communicated*) needed to solve problem
  - processors have *unlimited internal resources* (i.e., can compute anything)
  - For today, will assume each processor deterministic

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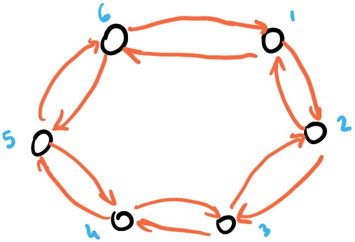

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- **Theorem:** all processors identical (same set of states and transition functions) and deterministic then it is *impossible* to elect a leader!
- To show this, simply look at execution and check that all processors will always be at identical states.

Leader Election: Algorithm

- Let's assume that each processor also has a unique ID (UID)
- But they don't know size of the network (i.e. n)

using randomness, can pick unique ID
with high probability

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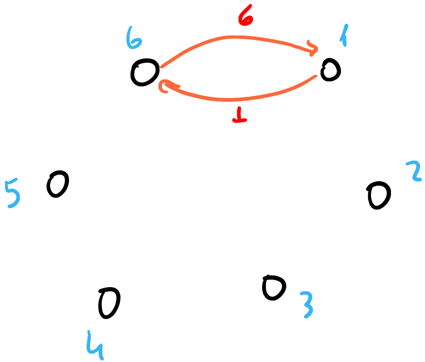

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    - if it is bigger, pass it on
    - if smaller, discard
    - equal $\Rightarrow$ processor declares itself leader
    - leader then notifies everyone else (by message relaying in network)

Leader : the one with largest UID

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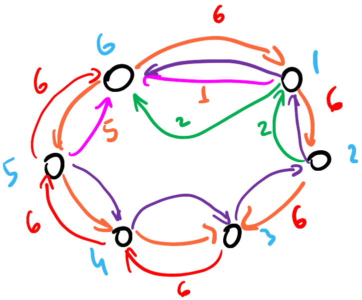

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- Can reduce communication to $O(n \log n)$ by successively doubling (see reference)

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- Model: synchronous model, arbitrary number of message failures.
- **Input**: Each processor has one bit of input. 1 (attack) or 0 (don't attack)
- **Output**: all should have $\overbrace{\textit{same decision bit } b}^{\text{agreement}}$ satisfying *weak validity*.[1]
  - if all processors start with bit 0, then 0 is only allowed decision
  - if all start with 1 and *all messages successfully delivered*, then 1 is the only allowed decision.

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Consensus Problem - Byzantine Failures

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worst case

  1. *Agreement*: same decision bit b
  2. *Weak Validity*: if all *non-faulty processors* ~~[illegible]~~ start with bit a, then b must be equal to a.

OBS: non-faulty processors are always able to correctly communicate

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  1. *Agreement*: same decision bit b
  2. *Weak Validity*: if all *non-faulty processors* processors start with bit a, then b must be equal to a.
- Complexity measures: *number of rounds* & *communication* (# messages exchanged in bit-size).

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n = #generals (size of our network) f = # faulty nodes

Example: $n = 3$ $f = 1$

P_1, P_2 honest P_3 faulty

inputs: $x_1 = 1$, $x_2 = 0$, $x_3 = 0$

P_3 send 1 to P_1 $\longrightarrow$ P_1 sees 1 0 1 attack

P_3 send 0 to P_2 $\longrightarrow$ P_2 sees 1 0 0 don't attack

Violated the agreement constraint!

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- At each round, each vertex broadcasts its knowledge
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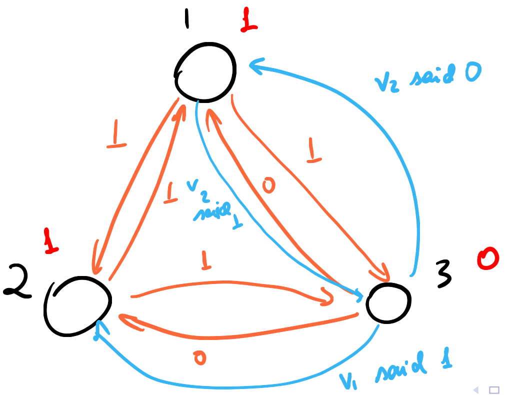
- At each round, each vertex broadcasts its knowledge
- After a number of rounds, everyone must make a decision
- Does this work?
- How many rounds do we need?
- How many Byzantine failures can it tolerate?

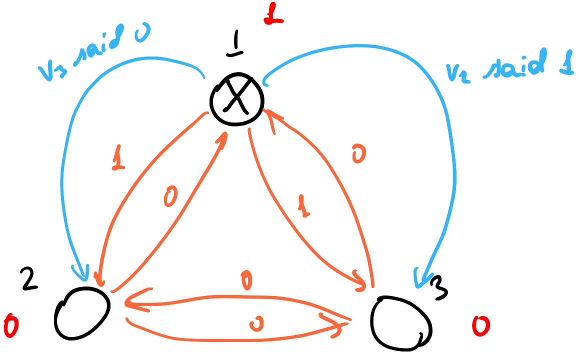
Byzantine Consensus - Bad Example

- 3 vertices $\{v_1, v_2, v_3\}$, 1 faulty vertex $n=3$, $f=1$ #rounds $=2$
- Scenario 1: $\boxed{v_1, v_2 \text{ good with value } 1,}$ v_3 faulty with value 0
  1. Round 1: all vertices truthful
  2. Round 2: v_3 lies to v_1, saying that v_2 said 0, all other communications truthful
  3. Validity $\Rightarrow v_1, v_2$ must decide 1



Byzantine Consensus - Bad Example

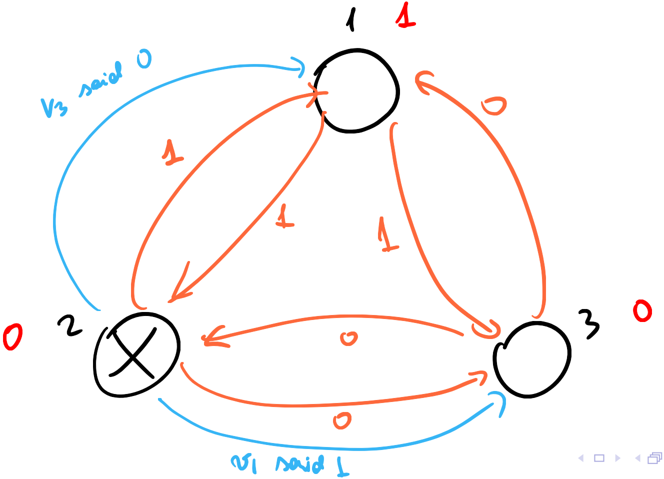
- 3 vertices $\{v_1, v_2, v_3\}$, 1 faulty vertex
- Scenario 2: v_2, v_3 good with value 0, v_1 faulty with value 1
    1. Round 1: all vertices truthful
    2. Round 2: v_1 lies to v_3, saying that v_2 said 1, all other communications truthful
    3. Validity $\Rightarrow$ v_2, v_3 must decide 0



Byzantine Consensus - Bad Example

- 3 vertices $\{v_1, v_2, v_3\}$, 1 faulty vertex
- Scenario 3: $\boxed{v_1, v_3 \text{ good with values } 1, 0}$ (resp.), v_2 faulty with value 0
  1. Round 1: v_2 tells v_1 its value is 1, tells v_3 its value is 0
  2. Round 2: all truthful

all they need to do is to agree



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- Scenario 1: v_1, v_2 good with value 1, v_3 faulty with value 0
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- Scenario 3: v_1, v_3 good with values $1, 0$ (resp.), v_2 faulty with value 0
  1. Round 1: v_2 tells v_1 its value is 1, tells v_3 its value is 0
  2. Round 2: all truthful
- Scenarios 1 and 3 identical to v_1, so it must return 1 (validity)
- Scenarios 2 and 3 identical to v_3, so it must return 0 (validity)
- Contradicts *agreement* in Scenario 3!

no algorithm with the gossiping in 2 rounds can work!

Byzantine Consensus - Algorithm

- Assumption:[2] $n > 3f$ (number of bad vertices $<$ third total vertices)

(in bad example $n = 3f$)

[2] It turns out that $n \leq 3f \Rightarrow$ *no algorithm* can reach consensus!

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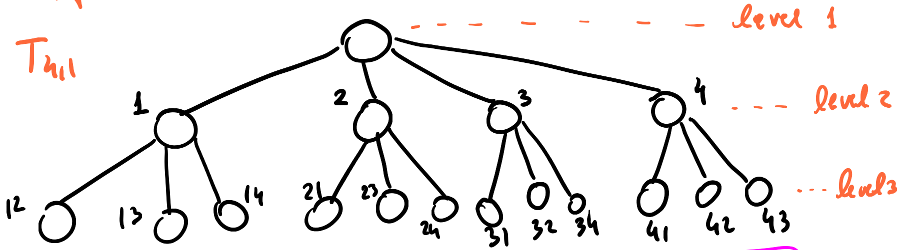
- Data structure: *Exponential Information Gathering* (EIG) tree $T_{n,f}$
  - Depth: $f+1$ (so $f+2$ node levels)
  - Each tree node at level $k+1$ labeled by string $i_1 i_2 \cdots i_k$ ($i_a \neq i_b$)

$n=4$, $f=1$

$T_{4,1}$



data structure

$\boxed{\text{Q: what does it contain?}}$

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  - Each tree node at level $k + 1$ labeled by string $i_1 i_2 \cdots i_k$ $(i_a \neq i_b)$
  - Node $i_1 i_2 \cdots i_k$ will store value v if the following happens: i_k told you that i_{k-1} told i_k that i_{k-2} told i_{k-1} ... that i_1 told i_2 that its initial value was v

$$\overset{V}{\overset{v}{i_1}} \xrightarrow{\alpha_1} \overset{v}{\overset{v}{i_2}} \xrightarrow{\alpha_2} \overset{w}{\overset{v}{i_3}} \xrightarrow{\alpha_3} \overset{w}{\overset{v}{i_4}} \longrightarrow \cdots \longrightarrow i_{k-1} \longrightarrow \overset{w}{\overset{v}{i_k}} \xrightarrow{v} \text{you}$$

idea: this tree is keeping track of common communicated information!

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Byzantine Consensus - EIG Tree

Byzantine Consensus - EIG Algorithm

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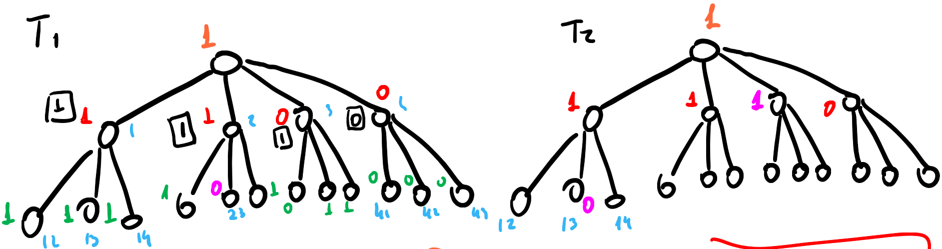
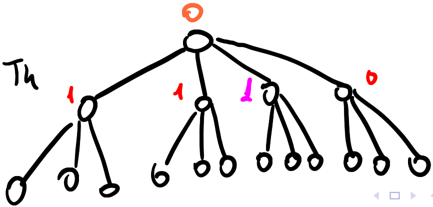
   - At round r, each vertex sends the values of level r of its EIG tree
   - Each vertex decorates values of its $(r + 1)^{th}$ level with values from messages
3. After $f + 1$ rounds, redecorate tree bottom-up, taking strict majority of children (otherwise set value of tree node to $\perp$)

EIG Algorithm - Example

- $n = 4$, $f = 1$
- p_3 is faulty, initial values are $p_1 = p_2 = 1$, $p_3 = p_4 = 0$
- round 1: p_3 lies to p_2 and p_4
- round 2: p_3 lies to p_2 about p_1 and lies to p_1 about p_2

wrong values in pink





Check: in this case all nodes will output 1

EIG Algorithm - Analysis

Lemma (Consistency of Non-Faulty Messages)

If i, j, k are non-faulty, then $T_i(x) = T_j(x)$ whenever label x ends with k.

$$x = x_{i_1} x_{i_2} \cdots \underbrace{x_{i(\ell+1)}}_{=k}$$

non-faulty

$$x = 1\,2\,3\,4\,k$$

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \xrightarrow{\boxed{v}} k \xrightarrow{v}$$

honest

EIG Algorithm - Analysis

Lemma (Consistency of Upwards Relabeling)

If label x ends with non-faulty process, then for any two non-faulty processors i, j the new values of $T_i(x)$ and $T_j(x)$ are the same.

EIG Algorithm - Analysis

Lemma (Consistency of Upwards Relabeling)

If label x ends with non-faulty process, then for any two non-faulty processors i, j the new values of $T_i(x)$ and $T_j(x)$ are the same.

- Base case: if x is the label of leaf, previous lemma handles it.

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- Base case: if x is the label of leaf, previous lemma handles it.
- Inductive step: $\boxed{|x| = t \leq f}$
  - By induction, if ℓ is a non-faulty element the new value of $T_i(\underline{x \circ \ell})$ is the same for any $i \in [n]$.

end with non faulty

OBS: if x is not leaf then know that $|x| \leq f$

because $T_{n,f}$ has $\underline{f+2}$ layers and $t+1$ layer has

t symbols $x_{i_1} x_{i_2} \cdots x_{i_t}$

$\boxed{x}$ not leaf $\Rightarrow$ x in layer $t+1 < f+2$

$\Rightarrow t \leq f$

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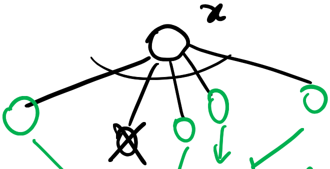
  - So label x has same labeled children across trees (if x_ℓ honest)



same values across all good trees Ti

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$$\boxed{= n - t} > 3f - f = \boxed{2f}$$

x at t+1 level

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  - At most f are faulty. By taking majority, we get that new values $T_i(x) = T_j(x)$

because > half of x's children agree

EIG Algorithm - Analysis

So far we have managed to prove:

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   - proof analogous to the proof of previous lemma
   - just note that all values will be b, as it is value being propagated by non-faulty nodes
3. *Agreement*: all nodes must agree on same value
   - By first lemma, all values in the leaves x are consistent across processors so long as x ends on a non-faulty process
   - By second lemma, majority will cause all values in nodes from level r ending in non-faulty nodes to be *the same* across processors
   - Induction and $n > 3f$ ensures that labels in level 1 will look the same on non-faulty nodes $\Rightarrow$ agreement

Conclusion

- Today we learned about distributed computation
- It is cool
- Widely used in practice
    - Cryptocurrencies - all of them need to solve Byzantine Agreement!

      Happening at UW: Sergey Gorbunov (involved with Algorand)
    - Other peer-to-peer systems
    - Multi-core programming

      Happening at UW: Trevor Brown
    - Biology (social insect colony algorithms)
    - many more...
- Learned an (inefficient) algorithm for Byzantine Agreement (check out the more efficient one in [Attiya and Welch 2004])

Acknowledgement

- Lecture based largely on:
  - Nancy Lynch's 6.852 Fall 2015 course - lectures 1 and 6
  - Lecture 1

    `https://learning-modules.mit.edu/service/materials/groups/103042/files/271154f5-ea0f-41a0-9ed9-6f83a5222d8b/link?errorRedirect=%2Fmaterials%2Findex.html&download=true`
  - Lecture 6

    `https://learning-modules.mit.edu/service/materials/groups/103042/files/95f71f5e-7791-4a1a-aeb5-e3d97afb167f/link?errorRedirect=%2Fmaterials%2Findex.html&download=true`

References I

Attiya, H. and Welch, J., 2004.
Distributed computing: fundamentals, simulations, and advanced topics (Vol. 19).
John Wiley & Sons.