Lecture 20: Online Algorithms & k-server

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Overview

- Online Algorithms: Randomized Lower Bounds
- k-server on a line
- Conclusion
- Acknowledgements

Competitive Analysis

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A deterministic online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

$$C_A(s) \leq k \cdot C_{opt}(s) + O(1)$$

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Definition (Randomized Competitive Ratio)

A randomized online algorithm A has *competitive ratio* k (aka k-competitive) if for all inputs s, we have:

worst-case

expectation over the randomness of your algorithm

$$\mathbb{E}[C_A(s)] \leq k \cdot C_{opt}(s).$$

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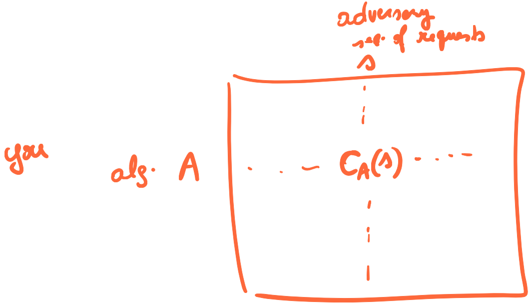

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LP duality applied to games

Theorem (Yao's minimax principle)

If for some input distribution, no deterministic algorithm is k-competitive, then no randomized algorithm is k-competitive!

randomized algorithm $\leftrightarrow$ mixed strategy for you

probability distribution over the possible sequences of requests $\leftrightarrow$ mixed strategy for adversary

(how to choose sequence of requests at random)

From Lect. 12

if you use $\underbrace{\text{randomized algorithm}}_{\text{mixed strat.}}$ then adversary can (and will) use $\underbrace{\text{pure strategy}}_{\text{worst possible req.}}$ as best response

competitiveness for rand. alg.

= if adversary uses $\underbrace{\text{prob. distribution over sequences}}_{\text{mixed strat.}}$ then you can use det. algorithm

Lower Bound - Randomized Paging Algorithms

1. Setting: $k+1$ distinct pages, cache of size k, n requests

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Overall $\boxed{\mathop{E}_{s}\left[C_A(s)\right] = \frac{n}{k+1}}$

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$$\mathbb{E}_s\left[C_{opt}(s)\right] = \frac{n}{\Theta(k \log k)}$$

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Lower Bound - Randomized Paging Algorithms

1. Setting: $k+1$ distinct pages, cache of size k, n requests
2. Distribution of inputs: *uniform distribution* $\frac{\mathbb{E}(C_A)}{\mathbb{E}[C_{opt}]}$ = competitive ratio
3. Equivalently: each page has probability $\frac{1}{k+1}$ of being chosen
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Theorem

Any randomized algorithm for paging with k pages is $\Omega(\log k)$-competitive!

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Practice problem: prove this!

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- Paging is special case of this problem (points of simplex)
- Today's Simplification: assume X is a *line*. Think $X = \mathbb{R}$

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$s = \frac{3}{4}, \frac{5}{4}, \frac{3}{4}, \frac{5}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$

Attempt 1: Greedy

Over n requests $\quad C_{greedy}(s) = \frac{1}{4} + \frac{1}{2}(n-1)$

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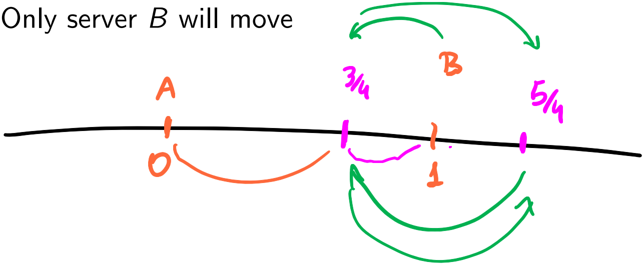

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6. Best strategy: put A on $3/4$, B on $5/4$

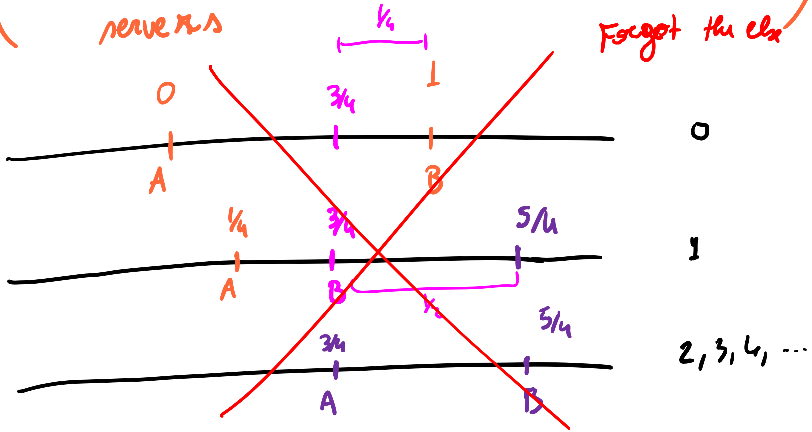
$C_{opt}(s) = \frac{3}{4} + \frac{1}{4} = 1$

Attempt 2: Double Coverage (DC)

- If request falls between two servers, move both towards request at same rate until one reaches it

(Simplification: never query midpoint between servers)

Forget the els



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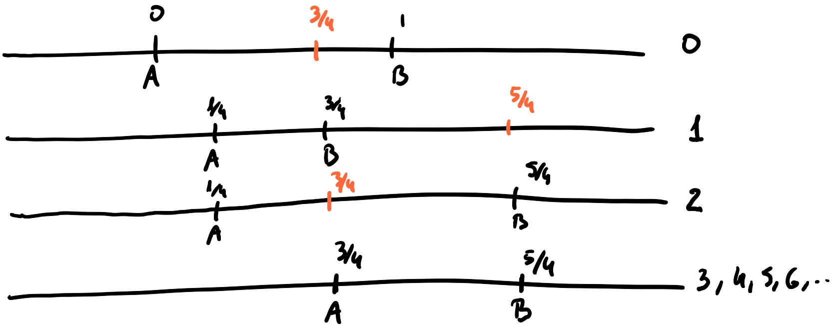

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5. Potential Function:
   - match each server from DC to a server of OPT
   - track changes as requests come

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- Total ammortized cost:

$$\begin{aligned} \sum_{t=1}^{n} \gamma_t &= \sum_{t=1}^{n} \left(c_t + \Phi_t - \Phi_{t-1} \right) \\ &= \underbrace{\Phi_n}_{\text{final potential}} - \underbrace{\Phi_0}_{\text{initial potential}} + \underbrace{\sum_{t=1}^{n} c_t}_{\text{total true cost}} \end{aligned}$$

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$$\sum_{t=1}^{n} \gamma_t = \sum_{t=1}^{n} c_t + \Phi_t - \Phi_{t-1}$$

$$= \cancel{\Phi_n} - \Phi_0 + \sum_{t=1}^{n} c_t \geq -\Phi_0 + \sum_{t=1}^{n} c_t$$

- If potential function is always *non-negative* $\Phi_t \geq 0 \quad \forall t$

and if $\Phi_0 = 0$

$$\sum_{t=1}^{n} c_t \leq \cancel{\Phi_0} + \sum_{t=1}^{n} \gamma_t$$

DC Analysis - Potential Function

Main idea: have the *ammortized cost* per request be (a multiple of) the cost of OPT, while the actual cost is the cost of DC.

if $\gamma_t = \alpha \cdot C_{opt}(\text{time } t)$

$e_t = C_A(\text{time } t)$

$$\sum c_t = C_A(\sigma) \leq \alpha \cdot C_{opt}$$

$$\alpha \cdot C_{opt} = \sum \gamma_t$$

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- Consider the state of DC and of OPT at time t

assume they start at the same state

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- Consider the state of DC and of OPT at time t
- Let M_t be cost of minimum cost matching between DC's servers and OPT servers
- Let S_t be sum of pairwise distances of DC's servers

initially have all of them at same pt

$M_0 = 0$, $S_0 = 0$

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- Our potential function will be

$$\Phi_t = \underbrace{k \cdot M_t}_{\text{min cost matching between servers of DC and servers of OPT}} + \underbrace{S_t}_{\text{sum of pairwise distances of DC's servers}}$$

$$\sum_{i \neq j} |x_i - x_j|$$

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- Use Amortized Analysis to compute amortized cost of DC
- Break requests into two parts:
  - First account for OPT move
  - Then account for DC move

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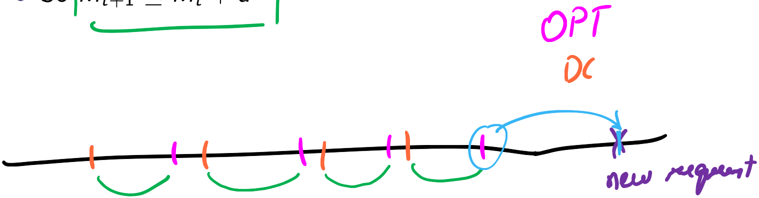

$M_{t+1} \le$ old matching now has cost $\le \underbrace{\text{old cost}}_{M_t} + d$

$\Rightarrow$ there exists a matching of cost $\le M_t + d$

M_{t+1} is a minimum cost matching

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   - So $M_{t+1} \leq M_t + d$
   - Thus potential increased (so far) by $\Phi_{t+1} - \Phi_t \leq k \cdot d$

$$\Phi_{t+1} - \Phi_t = k(M_{t+1} - M_t) \leq k \cdot d$$

(we still have not accounted for DC moves)

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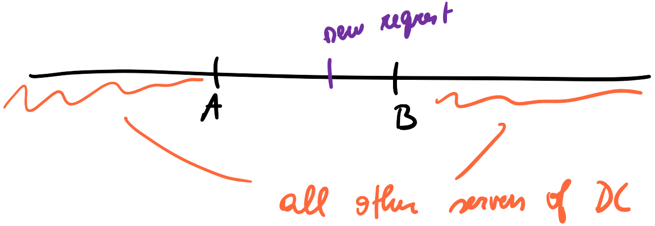

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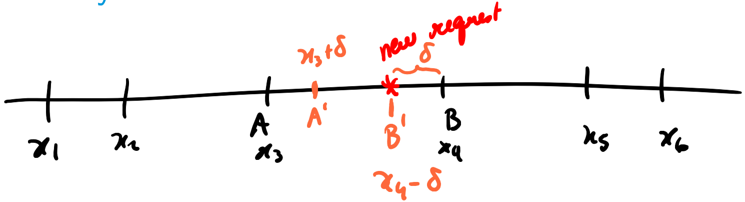

2. DC moves
    1. The request falls between two servers A and B. Say that B is taken to the location requested.
        - Both servers move a distance δ.
        - Thus pairwise distances decrease by 2δ (because they are in a line)

$$\sum \text{pairwise distances} = \sum_i \left(\text{distances from } x_i\right)$$

$$d(A', B') = d(A, B) - 2\delta$$

this does not get cancelled

DC Analysis - Potential Function

2. DC moves
   1. The request falls between two servers A and B. Say that B is taken to the location requested.
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$$S_{t+1} = S_t - 2\delta$$

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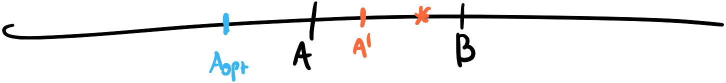
DC Analysis - Potential Function



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      - Potential Change: $\Phi_{t+1} - \Phi_t \leq k \cdot 0 - 2 \cdot \delta = -2 \cdot \delta$

$$\leq k \underbrace{(M_{t+1} - M_t)}_{\leq 0} + \underbrace{(S_{t+1} - S_t)}_{= -2\delta}$$

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DC Analysis - Potential Function

$$\gamma_{t+1} = c_{t+1} + \Delta\Phi$$
$$= -2\delta + \Delta\Phi \leq -2\delta$$

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      - Ammortized cost of DC: $\gamma_{t+1} \leq 2\delta - 2\delta = 0$

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1. OPT moves distance d
   - Ammortized cost of DC: $\gamma_t \leq k \cdot d$
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   2. Only one server, say A, moves (request outside the border)

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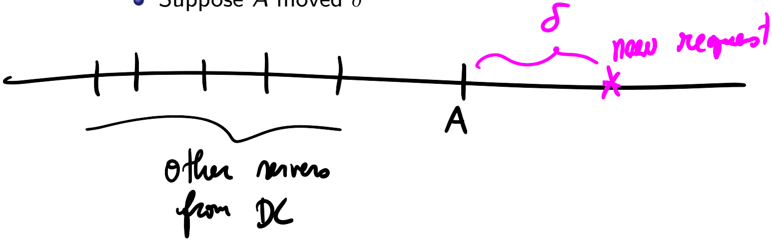

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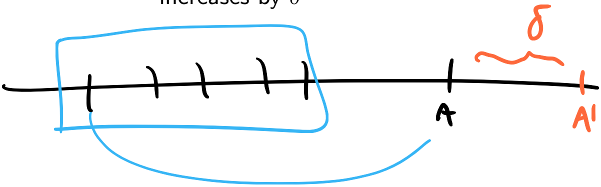
      - Each pairwise distance (A, B) (where B is another of DC's servers) increases by δ



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      - Total distance increased: $S_{t+1} - S_t \leq (k-1) \cdot \delta$

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        - Total distance increased: $S_{t+1} - S_t \leq (k-1) \cdot \delta$
        - Change in potential:

$$\Delta\Phi \leq \underline{-k \cdot \delta} + \underline{(k-1) \cdot \delta} = -\delta$$

$$= k(M_{t+1} - M_t) + (S_{t+1} - S_t)$$

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$c_t \geq \Delta\Phi$

      - Real cost incurred by DC: $c_{t+1} = \delta$
      - Ammortized cost at this step: $\gamma_{t+1} = \leq \delta - \delta = 0$

$= c_t + \Delta\Phi$

DC Analysis - Wrapping Up

1. OPT moves distance d
   - Ammortized cost of DC: $\gamma_t \leq k \cdot d$
2. DC moves
   1. The request falls between two servers.
      - Ammortized cost of DC: $\gamma_t \leq 0$
   2. Only one server moves (request outside the border)
      - Ammortized cost at this step: $\gamma_t \leq \delta - \delta = 0$

$\boxed{\gamma_t \leq 0}$

Total ammortized cost per request is the sum of
the ammortized cost from 1 (OPT)
plus " " from 2

$\therefore$ at each request $\gamma_t^{final} \leq k \cdot d_{OPT,t}$

DC Analysis - Wrapping Up

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- By our potential function inequality, we have:

$$\underbrace{\sum_{t=1}^{n} c_t}_{C_{DC}} \leq \cancelto{0}{\Phi_0} + \underbrace{\sum_{t=1}^{n} \gamma_t}_{\text{total ammortized cost}} \leq \sum_{t=1}^{n} k \cdot d_{OPT,t}$$

$$k \cdot C_{opt} = \sum_{t=1}^{n} k \cdot d_{OPT,t}$$

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- By our potential function inequality, we have:

$$\sum_{t=1}^{n} c_t \leq \Phi_0 + \sum_{t=1}^{n} \gamma_t$$

- Since $\gamma_t \leq k \cdot d$ whenever OPT moves d, and $\gamma_t \leq 0$ when OPT doesn't move, we have that $\sum_t \gamma_t \leq k \cdot C_{opt}$

DC Analysis - Wrapping Up

DC is k-competitive

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- By our potential function inequality, we have:

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- Since $\gamma_t \leq k \cdot d$ whenever OPT moves d, and $\gamma_t \leq 0$ when OPT doesn't move, we have that $\sum_t \gamma_t \leq k \cdot C_{opt}$
- Since Φ_0 is the initial state, we can regard it as constant (even 0, if require that servers start at a certain place)

Conclusion

- Online algorithms are important for many applications, when we need to make decisions right when we receive the information.
- Applications in
  - Stock Market
  - Dating
  - Skiing
  - Caching
  - Machine Learning (regret minimization)
  - many more...
- *Competitive Analysis*: measures performance of our algorithm against best algorithm that could *see into the future*
- Saw how to use *minimax theorem* in *Yao's principle* to prove lower bounds for randomized online algorithms.
- combined ammortized analysis in the online setting to solve k-server

Acknowledgement

- Lecture based largely on:
  - Lectures 18 & 20 of Karger's 6.854 Fall 2004 algorithms course
  - [Motwani & Raghavan 2007, Chapter 13]
- See Karger's Lecture 18 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s23-onlineRandomLb.pdf`
- See Karger's Lecture 20 notes at

  `http://courses.csail.mit.edu/6.854/06/scribe/s24-paging.pdf`

References I

Motwani, Rajeev and Raghavan, Prabhakar (2007)
Randomized Algorithms